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Emmanuele Di Benedetto and Michel Pierre

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ABSTRACT

For an m-accretive operator A in a Banach space X, we investigate the invariance of the solution of $\frac{d}{dt}(u + \lambda Au) + Au \ni 0$ with respect to a convex cone, under the assumption that the resolvents of A leave invariant the cone.

If in particular X is a function space and above represents a partial differential equation, necessary and sufficient conditions are given on the boundary data to insure the nonnegativity of the solution.

AMS(MOS) Subject Classification: 35K55, 35K70.

Key Words: m-accretive operators, maximum principle, invariant convex, filtration problem.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

d/dt (MU - LAMBDA DELTA MU) - DELTA MU = 0

THIS REPORT

Consider the equation $\frac{d}{dt}(u - \lambda\Delta u) - \Delta u = 0$ in a cylindrical domain. Unlike the heat equation, the positivity of the boundary data is not sufficient to insure that the solution is nonnegative. It is desirable to identify those boundary data for which the above property is true. One reason is that, since the above equation is a model for heat conduction and for fluid flow in fractured porous media, it is of interest to locate those boundary data that make the correspondent physical process meaningful.

In this paper several boundary value problems associated with the above equation are studied and necessary and sufficient conditions on the data are given to insure the nonnegativity of the solution.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE MAXIMUM PRINCIPLE FOR
PSEUDOPARABOLIC EQUATIONS

Emmanuele Di Benedetto and Michel Pierre

INTRODUCTION

Let A be an m-accretive operator in a Banach space X; it is well-known that if its resolvents J_λ satisfy:

(1) $$\forall\lambda > 0 \qquad J_\lambda C \subset C \quad ,$$

for a given closed convex set C of X, then the solution of:

(P) $$\frac{du}{dt} + Au \ni 0 \quad , \quad u(0) = u_0 \quad ,$$

satisfies:

(2) $$u_0 \in C \Rightarrow \forall t \qquad u(t) \in C \quad .$$

Here, we study the same problem for the associated pseudoparabolic equations:

(PP) $$\frac{d}{dt}(u + \lambda Au) + Au \ni 0 \quad , \quad u(0) = u_0 \qquad (\lambda > 0) \quad .$$

or, more generally, if $A(t)$ is a family of m-accretive operators satisfying (1):

$(PP)_t$ $$\frac{d}{dt}(u + \lambda A(t)u) + A(t)u \ni 0 \quad , \quad u(0) = u_0 \quad .$$

In order to exhibit a concrete situation, we remark that equation $(PP)_t$ contains as a particular case, the following problem:

(E) $$\begin{cases} \frac{\partial}{\partial t}(u - \lambda\Delta u) - \Delta u = 0 \quad \text{in} \quad \Omega \times [0,T[\\ u(t)\Big|_{\partial\Omega} = g(t) \quad , \quad u(0) = u_0 \quad , \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^n$. This equation can be assumed as a model for diffusion of fluids in fractured porous media (see [2]), or as a model for heat-conduction involving a thermodynamic temperature $\theta = u - \lambda\Delta u$ and a conductive temperature u (see [11], [23]). Moreover, when λ is small, it is an approximation of the classical heat equation (i.e. (E) when $\lambda = 0$) (see [19], [22]).

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It is well-known that in the latter case $(\lambda = 0)$:

(3) $$(u_0 \geq 0 \quad , \quad \forall t \quad g(t) \geq 0) \Rightarrow (\forall t \quad u(t) \geq 0) \quad ,$$

which is the property (2) when C is the cone of nonnegative functions in some $L^p(\Omega)$. But is (3) true for the equation (E) when $\lambda > 0$?

In [23], Ting proved that, if <u>$g(t) \equiv 0$</u>, the solution u of (E) satisfies:

(4) $$0 \leq u_0 \Rightarrow \forall t \quad 0 \leq u(t) \quad .$$

In fact this result is a particular case of a more general situation. Namely if $A(t) \equiv A$ is a <u>linear</u> and <u>time-independent</u> m-accretive operator satisfying (1), the solution of (PP) satisfies (2) (cf. proposition 1.2).

But when A is nonlinear or depends on time, this result is no longer true. For instance Rundell and Stecher noticed in [20] that the mere nonnegativity of u_0 and g is not sufficient to insure a nonnegative solution for (E). This shows that extra assumptions on $A(t)$ are needed to obtain the invariance property (2) for $(PP)_t$ or even (PP). The purpose of this paper is to give some results in this direction together with related questions.

Our study is divided in three parts.

The first section contains abstract results. For example, using the fact that, if A satisfies (1), its Yosida-approximations also do, we easily show the linear result indicated above and the following general property: if u is the solution of $(PP)_t$ where $A(t)$ verify (1)

$$u_0 + A(0)u_0 \in C \Rightarrow \forall t \quad u(t) + A(t)u(t) \in C \quad ,$$

and then $u(t) \in C \quad \forall t$.

In the second paragraph, we study problems of type (E). We show that the nonnegativity of the data is preserved in (E) if g does not decay too rapidly. More precisely, the solution u of (E) satisfies:

(5) $$u_0 \geq 0 \Rightarrow \forall t \quad u(t) \geq 0 \quad ,$$

if and only if:

$$\forall t \qquad g(t) \geq e^{-t/\lambda} g(0) \quad .$$

The case of Neumann boundary data is also considered.

The third paragraph studies the equation (PP) for "classical" nonlinear operators A in $L^p(\Omega)$ and the results are quite surprising. Assuming that

$$u_0 \geq 0 \Rightarrow \forall \lambda > 0 \qquad J_\lambda^A u_0 \geq 0 \quad ,$$

the fact that the associated equation (PP) satisfies the maximum principle (5) depends on the nature of the nonlinearity of A. Let us summarize some results. Let β denote a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta 0$ and let us consider the following (formal) equations:

$$(E_1) \qquad \frac{\partial}{\partial t}(u - \lambda\Delta\beta u) - \Delta\beta u \ni 0 \quad , \quad (\beta u)(t)\Big|_{\partial\Omega} = 0 \quad , \quad u(0) = u_0 \quad .$$

$$(E_2) \qquad \frac{\partial}{\partial t}(u - \lambda\Delta u) - \Delta u = 0 \quad , \quad -\frac{\partial u}{\partial n} \in \beta(u) \text{ on } \partial\Omega \quad , \quad u(0) = u_0 \quad .$$

$$(E_3) \qquad \frac{\partial}{\partial t}(u + \lambda(-\Delta u + \beta u)) - \Delta u + \beta u \ni 0 \quad , \quad u\Big|_{\partial\Omega} = 0 \quad , \quad u(0) = u_0 \quad .$$

Then:

(E_1) satisfies (5) for <u>any</u> β.

(E_2) satisfies (5) <u>if and only if</u> $\left\{ r \in \,]0,\infty[\, \cap D(\beta) \to \frac{\beta^0(r)}{r} \right\}$ is nondecreasing.

(E_3) satisfies (5) if $D(\beta) \subset [0,\infty[$ or $\left\{ r \in \,]0,\infty[\, \cap D(\beta) \to \frac{\beta^0(r)}{r} + \frac{1}{\lambda} \ln r \right\}$ is nondecreasing and the latter condition is necessary if β is regular.

1. ABSTRACT RESULTS

In this section we denote by X a Banach space with the norm $|\cdot|$ and T a positive number. As usual $L^1(0,T;X)$ (resp. $C([0,T];X)$) is the space of integrable (resp. continuous) functions from $[0,T]$ into X and

$$W^{1,1}(0,T;X) = \{u \in L^1(0,T;X) \; ; \; \frac{du}{dt} \in L^1(0,T;X)\}$$

(see [7] appendix for more details about this space).

For any $t \in [0,T]$, we will denote by $A(t)$ a (possibly multi-valued) operator in X, i.e. an application from X into 2^X with domain $D(A(t)) = \{x \in X \; ; \; A(t)x \neq \emptyset\}$ and range $R(A(t)) = \bigcup_{x \in X} A(t)x$ (we identify $A(t)$ with its graph in $X \times X$). We will study the associated pseudoparabolic equation

$$(PP)_t \qquad \frac{d}{dt}(u(t) + \lambda A(t)u(t)) + A(t)u(t) \ni f(t) \quad , \quad u(0) = u_0 \quad ,$$

where $u_0 \in X$, $f \in L^1(0,T;X)$ and $\lambda > 0$.

First we make precise the meaning of solution of $(PP)_t$ and the assumptions on $A(t)$ that insure its existence.

A) Existence results

As it is the case for the associated parabolic equation:

$$(P)_t \qquad \frac{d}{dt}u(t) + A(t)u(t) \ni f(t) \quad , \quad u(0) = u_0 \quad ,$$

and as it has already been remarked and used in [19], [18], [22], it is natural to assume that the operators $A(t)$ are m-accretive (we refer e.g. to [12], [15] or [3] for more details about this notion--let us just recall that an operator A on X is said m-accretive if, for any $\lambda > 0$, $I + \lambda A$ is onto and $J_\lambda = (I + \lambda A)^{-1}$ is a nonexpansive mapping from X into itself--).

In fact, under this assumption, $(PP)_t$ is far easier to solve than $(P)_t$ and even has "strong" solutions which generally is not the case for $(P)_t$. However a difficulty arises for the uniqueness when the operators are multivalued. If one interprets the equation $(PP)_t$ as: "there exist $w(t)$, $\hat{w}(t) \in A(t)u(t)$ such that

$$\frac{d}{dt}(u + \lambda w) + \hat{w} = f \quad , \quad u(0) = u_0" \quad ,$$

then the function u need not be uniquely determined even if $A(t) \equiv A$ is accretive, as shown by the simple example below. So we will require the selections w and $\hat{w}$ to be the same. Moreover it appears that the natural initial data is not $u_0 \in D(A(0))$, but an element $[u_0,w_0] \in A(0)$. In particular, even if one imposes $w = \hat{w}$, for u_0 fixed the solution generally varies with w_0.

<u>Example</u>: Let $X = \mathbb{R}$, $D(A) = [0,\infty[$, $A0 =]-\infty,1]$, $\forall x > 0$, $Ax = 1$. Then, the problem:

$$\begin{cases} \exists w(t) \quad , \quad \hat{w}(t) \in Au(t) \quad \text{with} \quad w(0) = \hat{w}(0) = 0 \quad , \\ \frac{d}{dt}(u + w) + \hat{w} = 2 \quad , \quad w(0) = 0 \quad , \end{cases}$$

admits the following two solutions:

$$u_1(t) = \begin{cases} 0 & \text{in} \quad [0,\ln 2] \\ t - \ln 2 & \text{in} \quad [\ln 2,\infty) \end{cases}$$

for the selections

$$w_1(t) = \hat{w}_1(t) = \begin{cases} 2(1 - e^{-t}) & \text{on} \quad [0,\ln 2] \\ 1 & \text{on} \quad [\ln 2,\infty) \quad , \end{cases}$$

and:

$$u_2(t) = \begin{cases} 0 & \text{on} \quad [0,\frac{2}{3}] \\ t - \frac{2}{3} & \text{on} \quad [\frac{2}{3},\infty) \end{cases}$$

for the selections

$$w_2(t) = \begin{cases} 2t - \frac{3}{4}t^2 & \text{on} \quad [0,\frac{2}{3}] \\ 1 & \text{on} \quad [\frac{2}{3},\infty) \end{cases} \quad , \quad \hat{w}_2(t) = \begin{cases} \frac{3}{2}t & \text{on} \quad [0,\frac{2}{3}] \\ 1 & \text{on} \quad [\frac{2}{3},\infty) \quad . \end{cases}$$

Moreover, the problem

$$\exists w(t) \in Au(t) \; ; \; \frac{d}{dt}(u + w) + w = 2 \quad , \quad u(0) = 0 \quad ,$$

admits the solution u_1 for $w(0) = 0$ and the solution $u_3(t) = t$ for $w_3(0) = 1$.

These remarks suggest the following definition of solution which is justified by the proposition 1.1 below.

DEFINITION. Given $[u_0,w_0] \in A(0)$ and $f \in L^1(0,T;X)$ we call solution of

$$(PP)_t \qquad \frac{d}{dt}(u + \lambda A(t)u) + A(t)u \ni f \quad , \quad u(0) = u_0 \quad , \quad A(0)u(0) \ni w_0 \quad ,$$

a function u of $C([0,T];X)$ such that:

$$(pp)_t \qquad \begin{cases} \exists w \in C([0,T];X) \text{ with } u + \lambda w \in W^{1,1}(0,T;X) \text{ and} \\ w(0) = w_0 \quad , \quad \forall t \in [0,T] \quad , \quad w(t) \in A(t)u(t) \\ u(0) = u_0 \quad , \quad \text{a.e. } t \in (0,T) \quad , \quad \frac{d}{dt}(u(t) + \lambda w(t)) + w(t) = f(t) \quad . \end{cases}$$

We denote by $J_\lambda(t) = (I + \lambda A(t))^{-1}$ the resolvent of $A(t)$ and state the following proposition.

PROPOSITION 1.1. Suppose $A(t)$ is m-accretive for any $t \in [0,T]$ and satisfies:

$$\forall x \in X \quad , \quad [t \to J_\lambda(t)x] \text{ is continuous on } [0,T] \quad .$$

Then, for any $[u_0,w_0] \in A(0)$ and $f \in L^1(0,T;X)$, there exists a unique solution of:

$$\frac{d}{dt}(u + \lambda A(t)u) + A(t)u \ni f \quad , \quad u(0) = u_0 \quad , \quad A(0)u(0) \ni w_0 \quad .$$

Moreover the solution w satisfying $(pp)_t$ is unique.

Remark 1.1. When the applications $[t \to J_\lambda(t)x]$ are only integrable (and a.e. defined), we can obtain a "solution" $u \in L^1(0,T;X)$, but the meaning of the initial conditions $u(0) = u_0$, $A(0)u(0) = w_0$ must then be understood in a weak sense to be precised.

Remark 1.2. In the case when $A(t) \equiv A$, $f \equiv 0$, the proposition above associates with any $[u_0,w_0] \in A$ and $t \in [0,\infty[$ a unique $[u(t),w(t)] \in A$. Hence the mapping

$$(u_0,w_0) \to S_\lambda(t)(u_0,w_0) = (u(t),w(t))$$

defines a semigroup of operators from A (identified with its graph) into itself, that is:

$$\mathcal{S}_\lambda(0) = I \quad , \quad [t \to \mathcal{S}_\lambda(t)(u_0,w_0)] \quad \text{is continuous}$$

$$\forall s,t \geq 0 \quad , \quad \mathcal{S}_\lambda(t+s) = \mathcal{S}_\lambda(t)\mathcal{S}_\lambda(s) \quad .$$

In fact, one can easily show that $\mathcal{S}_\lambda(t)$ can be extended in a group on $\mathbb{R}$.

Considering the distance defined on A by

$$d((u,w),(\hat{u},\hat{w})) = |u - \hat{u} + \lambda(w - \hat{w})| \quad ,$$

since A is accretive, $\mathcal{S}_\lambda(t)$ is a nonexpansive mapping on A for this metric.

Remark 1.3. If A is singlevalued, by setting $u(t) = S_\lambda(t)u_0$, we also define a continuous semigroup of operators from $D(A)$ into itself which converges to the semigroup generated by A (e.g. in the sense of [12]) when λ goes to 0 (see prop. 1.4). In general $S_\lambda(t)$ is not a contraction (see remark 1.5) and even cannot be continuously extended to $\overline{D(A)}$ (see corollary 3.1).

If A is linear, we can show that $S_\lambda(t)$ is a semigroup of contractions from $D(A)$ into itself. This is a consequence of the following linear properties:

$$\frac{d}{dt}(Au) = A\left(\frac{du}{dt}\right) \quad , \quad J_\lambda A = AJ_\lambda = A_\lambda$$

(we recall that if A is an accretive operator, its Yosida approximation is $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$).

For linear operators A we have:

PROPOSITION 1.2. Let A be a closed linear accretive operator in X, $u_0 \in D(A)$, $f \in L^1(0,T;X)$ with $f(t) \in D(J_\lambda)$ a.e.t; then the following statements are equivalent:

(i) $u \in C([0,T];X) \quad , \quad u + \lambda Au \in W^{1,1}(0,T;X)$

$$\frac{d}{dt}(u + \lambda Au) + Au = f \quad , \quad u(0) = u_0 \quad .$$

(ii) $u, Au \in W^{1,1}(0,T;X) \quad , \quad u'(t) \in D(A)$ a.e.t.

$$u' + \lambda Au' + Au = f \quad , \quad u(0) = u_0$$

(iii) $u \in W^{1,1}(0,T;X)$

$$u' + A_\lambda u = J_\lambda f \quad , \quad u(0) = u_0 \quad .$$

Moreover, if u and $\hat{u}$ are solutions of the above equations with data (u_0, f) and $(\hat{u}_0, \hat{f})$ respectively, then

$$\forall t \in [0,T] \quad , \quad |u(t) - \hat{u}(t)| \le |u_0 - \hat{u}_0| + \int_0^t |f(\sigma) - \hat{f}(\sigma)| d\sigma \quad .$$

Remark 1.4. *The property (iii) is of a particular interest. Together with lemma 1.1 below, it shows that, when $f = 0$, u and $u + \lambda Au$ are both solutions of the equation:*

$$u' + A_\lambda u = 0 \quad .$$

The last inequality (which is a direct consequence of (iii) and the accretivity of A_λ) gives a way to define a notion of solution for the equation (i) when $u_0 \in \overline{D(A)}$.

The proof of the proposition 1.1 rests upon the following lemma:

Lemma 1.1. Under the assumptions of the proposition 1.1, u is a solution of

$$(PP)_t \qquad \frac{d}{dt}(u + \lambda A(t)u) + A(t)u \ni f \quad , \quad u(0) = u_0 \quad , \quad A(0)u(0) \ni w_0 \quad ,$$

if and only if $u(t) = J_\lambda(t)v(t)$ where v is solution of

$$(P_\lambda) \qquad v \in W^{1,1}(0,T;X) \quad , \quad v_0 = u_0 + \lambda w_0 \quad , \quad \frac{dv}{dt} + A_\lambda(t)v(t) = f(t)$$

Proof of lemma 1.1.

If u is a solution of $(PP)_t$, setting $v(t) = u(t) + \lambda w(t)$ ($w(t)$ is the corresponding selection out of $A(t)u(t)$), we have $u(t) = J_\lambda(t)v(t)$ and $w(t) = \frac{v(t) - u(t)}{\lambda} = A_\lambda(t)v(t)$; hence v satisfies (P_λ).

Now let v be a solution of (P_λ) and $u(t) = J_\lambda(t)v(t)$; then $u \in C([0,T];X)$ by the assumptions on $J_\lambda(t)$ and the inequality:

$$\forall s,t \in [0,T] \quad , \quad |u(t) - u(s)| \le |J_\lambda(t)v(t) - J_\lambda(s)v(t)| + |v(t) - v(s)| \quad .$$

As $v(t) \in u(t) + \lambda A(t)u(t)$, there exists $w \in C([0,T];X)$ with $w(t) \in A(t)u(t)$ and $v = u + \lambda w$. Moreover

$$A_\lambda(t)v(t) = \frac{v(t) - u(t)}{\lambda} = w(t) \quad \text{for any} \quad t \in [0,T] \quad .$$

So u is a solution of $(PP)_t$.

Proof of Proposition 1.1.

By lemma 1.1, it is sufficient to prove existence and uniqueness of the solution of (P_λ). Since $A_\lambda(t)$ are Lipschitz-continuous on X and $t \to A_\lambda(t)x - f(t)$ integrable for any $x \in X$, the proposition follows from known results (see for example [7], example 1.3.2).

Proof of Proposition 1.2.

Suppose u satisfies (i); then, as $v = u + \lambda Au \in W^{1,1}(0,T;X)$ and since J_λ is linear and continuous on the closed set $D(J_\lambda)$, $u = J_\lambda v \in W^{1,1}(0,T;X)$ and $u'(t) = J_\lambda v'(t)$ a.e.t. By difference, $Au \in W^{1,1}(0,T;X)$ and since A is linear and closed $\frac{d}{dt}(Au) = Au'$. This proves (ii).

Clearly (ii) implies (i).

Now, if u satisfies (ii), by applying J_λ we obtain:

$$u'(t) + J_\lambda Au(t) = J_\lambda f(t) \quad .$$

From the linearity of A, $J_\lambda Ax = AJ_\lambda x$ for all $x \in D(A)$. Hence (ii) $\Rightarrow$ (iii).

Finally suppose u satisfies (iii); then:

$$\text{a.e.t.} \qquad u(t) + \lambda u'(t) - J_\lambda u(t) = J_\lambda f(t) \quad , \quad u(0) = u_0 \quad .$$

This proves that $(u + \lambda u')(t) \in D(A)$ and, by integration, that $u(t)$ and $u'(t) \in D(A)$. Therefore, by applying $I + \lambda A$ we obtain (ii).

The last inequality follows from (iii) by using the fact that A_λ is accretive.

B) Continuity results

The following results are a direct consequence of general facts about m-accretive operators and will be employed in section 3.

PROPOSITION 1.3. Let (A^n) be a sequence of m-accretive operators converging to the m-accretive operator A, i.e.

$$\forall \mu > 0 \quad , \quad \forall x \in X \quad , \quad \lim_{n\to\infty} J_\mu^{A^n} x = J_\mu^A x \quad .$$

Suppose $[u_0^n, w_0^n] \in A^n$ converges to $[u_0, w_0] \in A$ and f_n converges to f in $L^1(0,T;X)$. Then, the solution u^n of

$$\frac{d}{dt}(u^n + \lambda A^n u^n) + A^n u^n \ni f^n \quad , \quad u^n(0) = u_0^n \quad , \quad A^n u^n(0) \ni w_0^n \quad ,$$

converges in $C([0,T];X)$ to u solution of

$$\frac{d}{dt}(u + \lambda Au) + Au \ni f \quad , \quad u(0) = u_0 \quad , \quad Au(0) \ni w_0 \quad ,$$

and the corresponding sections w^n also converge in $C([0,T];X)$ to w.

PROPOSITION 1.4. Let A be an m-accretive operator in X, $[u_0^n, w_0^n] \in A$, $f^n \in L^1(0,T;X)$. Suppose f_n converge to f in $L^1(0,T;X)$ and there exists λ_n converging to 0 such that $u_0^n + \lambda_n w_0^n$ converges to $u_0 \in \overline{D(A)}$. Then when n goes to ∞, the solution of

$$\frac{d}{dt}(u^n + \lambda_n Au^n) + Au^n \ni f^n \quad , \quad u^n(0) = u_0^n \quad , \quad Au^n(0) \ni w_0^n \quad ,$$

converges in $C([0,T];X)$ to the integral solution (in the sense of [4]) of

$$\text{(P)} \qquad \frac{du}{dt} + Au \ni f \quad , \quad u(0) = u_0 \quad .$$

Remark 1.5. From the proposition 1.3, we can deduce that the pseudoparabolic semigroups associated with a singlevalued m-accretive operator (cf. remark 3) are not semigroups of contractions. To see this, let us consider A_ε the Yosida-approximations of a m-accretive operator A. Let $[u_0, w_0]$ and $[\hat{u}_0, \hat{w}_0] \in A$

with $w_0 \neq \hat{w}_0$ and let u_ε, $\hat{u}_\varepsilon$ be the solutions of

$$\frac{d}{dt}(u + A_\varepsilon u) + A_\varepsilon u = 0 \quad ,$$

with $u_\varepsilon(0) = u_0 + \varepsilon w_0$, $\hat{u}_\varepsilon(0) = u_0 + \varepsilon \hat{w}_0$ respectively. Since A_ε converges to A and $A_\varepsilon(u_0 + \varepsilon w_0) = w_0$, $A_\varepsilon(u_0 + \varepsilon \hat{w}_0) = \hat{w}_0$, by the proposition 1.3, u_ε and $\hat{u}_\varepsilon$ converge to the solutions of:

$$\frac{d}{dt}(u + Au) + Au = 0 \quad , \quad u(0) = u_0 \quad ,$$

with $Au(0) = w_0$ and $A\hat{u}(0) = \hat{w}_0$ respectively. Now u and $\hat{u}$ are in general different as shown by the previous example. Hence one cannot have:

$$\forall t > 0 \quad , \quad \forall \varepsilon > 0 \quad , \quad |u_\varepsilon(t) - \hat{u}_\varepsilon(t)| \leq |(u_0 + \varepsilon w_0) - (u_0 + \varepsilon \hat{w}_0)| = \varepsilon |w_0 - \hat{w}_0| \quad .$$

Proof of Proposition 1.3.

By a known result (see Benilan [4]), the solutions v^n of

$$\frac{dv^n}{dt} + (A^n)_\lambda v^n = f^n \quad , \quad v^n(0) = u_0^n + \lambda w_0^n \quad ,$$

converge in $C([0,T];X)$ to the solution of

$$\frac{dv}{dt} + A_\lambda v = f \quad , \quad v(0) = u_0 + \lambda w_0 \quad ;$$

By lemma 1.1, $u^n = J_\lambda^{A^n} v^n$ and $u = J_\lambda^A v$. Since $v([0,T])$ is compact and $J_\lambda^{A^n}$, J_λ^A are contractions, $J_\lambda^{A^n} v$ converges to $J_\lambda^A v$ in $C([0,T];X)$. Moreover we have:

$$|u^n(t) - u(t)| \leq |v^n(t) - v(t)| + |J_\lambda^{A^n} v(t) - J_\lambda^A v(t)| \quad .$$

Hence u^n converges to u in $C([0,T];X)$ and $w^n = \frac{v^n - u^n}{\lambda}$ converges also in $C([0,T];X)$ to $w = \frac{v - u}{\lambda}$.

Proof of Proposition 1.4.

Let v^n be the solution of

$$\frac{dv^n}{dt} + A_{\lambda_n} v^n \ni f^n \quad , \quad v^n(0) = u_0^n + \lambda_n w_0^n \quad .$$

By the quoted result (see [4], [3]), v^n converges in $C([0,T];X)$ to the integral solution (in the sense of [4]) of

$$\frac{du}{dt} + Au = f \quad , \quad u(0) = u_0 \quad .$$

Moreover:

$$|u^n(t) - u(t)| = |J_{\lambda_n} v^n(t) - u(t)| \leq |v^n(t) - u(t)| + |J_{\lambda_n} u(t) - u(t)| \quad ,$$

and $J_{\lambda_n} u - u$ converges to 0 in $C([0,T];X)$.

<u>Remark</u>. Continuity results similar to this proposition can be found in [22].

<u>C) Invariance properties for</u> $(PP)_t$

We denote by C a closed convex set in X.

<u>PROPOSITION 1.5</u>. Assume the hypothesis of proposition 1.1 and suppose that:

$$\text{(i)} \quad \text{a.e.t.} \qquad J_\lambda(t)C + \lambda f(t) \subset C$$

$$\text{(ii)} \quad u_0 + \lambda w_0 \in C \quad .$$

Then, $u(t) + \lambda w(t) \in C$ for all $t \in [0,T]$. If, moreover $J_\lambda(t)C \subset C$, then $u(t) \in C$ for all $t \in [0,T]$.

<u>PROPOSITION 1.6</u>. Let A be a <u>linear</u> m-accretive operator in X and u the solution of

$$\frac{d}{dt}(u + \lambda Au) + Au = f \quad , \quad u(0) = u_0 \quad ,$$

with $u_0 \in D(A)$ and $f \in L^1(0,T;X)$. Suppose:

$$\text{a.e.} t \in \,]0,T[\quad , \quad J_\lambda(C + \lambda f(t)) \subset C \quad .$$

Then:

$$(u_0 \in C) \Rightarrow (\forall t \in [0,T] \quad , \quad u(t) \in C) \quad .$$

<u>Remark 1.6</u>. To see the interest of the first proposition, let us suppose that $X = L^p(\Omega)$ for some $p \in [1,\infty]$ and some open set Ω in $\mathbb{R}^n$ and let

$C = \{u \in L^p(\Omega);\ u \geq 0\}$. Suppose that $A(t)$ is a family of m-accretive operators in $L^p(\Omega)$ satisfying the following maximum principle:

$$(u \geq 0) \Rightarrow (\forall \lambda \geq 0 \quad , \quad J_\lambda(t)u \geq 0) \quad .$$

Then, as an application of the proposition 1.5 we obtain that if u is the solution of

$$\frac{d}{dt}(u + \lambda A(t)u) + A(t)u \ni f \quad , \quad u(0) = u_0 \quad ,$$

then:

$$(u_0 + \lambda A(0)u_0 \geq 0 \quad , \quad f \geq 0) \Rightarrow (\forall t \quad , \quad u(t) + \lambda A(t)u(t) \geq 0)$$

and, therefore $u(t) \geq 0$ for all $t \in [0,T]$.

In the particular case when $A(t)$ are the operators associated with the equation (E) in the introduction, this result says that the thermodynamic temperature remains nonnegative for all $t > 0$ if it is so for $t = 0$. This was remarked in [23] for (E) when $g \equiv 0$. Above shows that this property is quite general.

Remark 1.7. A more interesting result is the following. Given a family of operators $A(t)$ satisfying:

$$(u \geq 0) \Rightarrow (J_\lambda(t)u \geq 0) \quad ,$$

what can be said about the positivity of the solutions of the associated pseudo-parabolic equation assuming that $u_0 \geq 0$ and $f \geq 0$?

The proposition 1.6 gives a first result; it tells that if $A(t)$ is linear, independent of time and satisfies the above maximum principle, then $u(t) \geq 0$ as soon as $u_0 \geq 0$ and $f \geq 0$.

But this is not necessarily true if $A(t)$ depends on time or if $A(t) \equiv A$ is not linear. The purpose of the next sections is precisely to study in particular cases what extra assumptions on $A(t)$ imply the nonnegativity of the solutions.

The main idea in the proof of both the propositions is the following (see [10]). If a closed convex set C is "invariant" by A, it is also invariant by A_λ, that is:

$$(\forall\lambda > 0 \quad , \quad (I + \lambda A)^{-1}C \subset C) \Rightarrow (\forall\mu > 0 \quad , \quad (I + \mu A_\lambda)^{-1}C \subset C) \quad .$$

Moreover also the semigroup generated by A_λ leaves C invariant. More precisely, we use the next lemma proved in [7] - Corollary 1.1.

<u>Lemma 1.2</u>. Let C be a closed convex set in X and, for any $t \in [0,T]$, let $J(t) : C \to C$ be a contraction such that $x \to J(t)x$ is integrable for any $x \in C$. Then, for any $v_0 \in C$ there exists a unique $v \in W^{1,1}(0,T;X)$ with <u>$v(t) \in C$</u> for all t satisfying:

$$v(0) = v_0 \quad , \quad \frac{dv}{dt} + \frac{v(t) - J(t)v(t)}{\lambda} = 0 \quad .$$

<u>Proof of Propositions 1.5 and 1.6</u>.

To prove 1.5 use lemma 1.1, and apply lemma 1.2 with $J(t)$ defined by $J(t)x = J_\lambda(t)x + \lambda f(t)$.

To prove 1.6 use the proposition 1.2, and apply lemma 1.2 to $v_0 = u_0$, and $J(t)$ defined by $J(t)x = J_\lambda(t)(x + \lambda f(t))$.

2. MAXIMUM PRINCIPLE FOR EQUATIONS OF TYPE (E)

In this section we describe some invariance properties of the solution of $(PP)_t$ with respect to convex cones, for some operators which are, roughly speaking "linear in the interior." The results of the next part A, will be applied to boundary value problems connected with equation (E), in part B.

A. Some general remarks

Let A be linear and m-accretive in X and let $G \in C([0,T];X)$ be given. For $t \in [0,T]$ define

$$D[A(t)] = \{u \in X \mid u - G(t) \in D(A)\}$$
$$A(t) = A(u - G(t)) - \frac{1}{\lambda} G(t) \quad .$$

Note that $A(t)$ is also m-accretive in X for all $t \in [0,T]$, hence by proposition 1.1, for any $u_0 \in D[A(0)]$ and $f \in L^1(0,T;X)$, there exists a unique solution of

$$\text{(E)} \qquad \frac{d}{dt}(u + \lambda A(t)u) + A(t)u = f \quad , \quad u(0) = u_0 \quad .$$

We will denote with $u_\lambda(\cdot,u_0,f)$ such a solution. Our next task is to derive a representation of $u_\lambda(\cdot,u_0,f)$ in terms of the operator A, which will be extensively used throughout this section. With $J_\lambda = (1 + \lambda A)^{-1}$, $\lambda > 0$, we denote the resolvents of A. A_λ is the Yoshida approximation of A and $S_\lambda(\cdot)$ denotes the linear contraction semigroup generated by $-A_\lambda$ in X.

<u>Lemma 2.1</u>. For all $u_0 \in D(A(0)) = D(A) + G(0)$ and $f \in L^1(0,T;X)$,

$$u_\lambda(t,u_0,f) = S_\lambda(t)u_0 + \Phi(t) + \int_0^t J_\lambda S_\lambda(t-s)f(s)ds \quad , \quad \forall t \in [0,T] \quad , \quad \text{where}$$

$$\Phi(t) = [G(t) - e^{-t/\lambda}G(0)] + \frac{1}{\lambda}\int_0^t J_\lambda S_\lambda(t-s)[G(s) - e^{-s/\lambda}G(0)]ds \quad .$$

<u>Remark 2.1</u>. This representation permits us to define a "generalized" solution of $(PP)_t$ <u>for all</u> $u_0 \in X$, since $S_\lambda(t)$ is a contraction defined in the whole space X.

Proof of Lemma 2.1.

Set $u(t) = v(t) + G(t)$. Then problem $(PP)_t$ can be rewritten as

$$\frac{d}{dt}(v + \lambda Av) + Av = \frac{1}{\lambda} G + f \quad , \quad v(0) = u_0 - G(0)$$

and by proposition 1.2

$$\frac{d}{dt} v + A_\lambda v = \frac{1}{\lambda} J_\lambda G + J_\lambda f \quad , \quad v(0) = u_0 - G(0) \; .$$

An easy verification shows that $w(t) = e^{-t/\lambda} G(0)$ is the unique solution of

$$\frac{d}{dt} w + A_\lambda w = -\frac{1}{\lambda} J_\lambda e^{-t/\lambda} G(0) \quad , \quad w(0) = G(0) \quad .$$

Adding the two previous equalities we have

$$\frac{d}{dt}(u - [G(t) - e^{-t/\lambda}G(0)]) + A_\lambda(u - [G(t) - e^{-t/\lambda}G(0)]) =$$

$$= \frac{1}{\lambda} J_\lambda [G(t) - e^{-t/\lambda}G(0)] + J_\lambda f \quad .$$

So that by the theory of linear contraction semigroups [16]

$$u(t) = S_\lambda(t)u_0 + [G(t) - e^{-t/\lambda}G(0)] + \frac{1}{\lambda}\int_0^t J_\lambda S_\lambda(t-s)[G(s) - e^{-s/\lambda}G(0)]ds +$$

$$+ \int_0^t J_\lambda S_\lambda(t-s)f(s)ds = S_\lambda(t)u_0 + \Phi(t) + \int_0^t J_\lambda S_\lambda(t-s)f(s)ds \quad .$$

This is the desired representation.

Now let C be a closed convex cone in X with vertex at zero, i.e. if $x,y \in C$, then $tx + sy \in C$, $\forall t,s \in \mathbb{R}^+$, and assume that A satisfies a "maximum principle" in the form

$$\text{(2.1)} \qquad J_\lambda C \subset C \qquad \forall \lambda \geq 0 \quad .$$

Then from a result of [10] we have also

$$\text{(2.2)} \qquad S_\lambda(t)C \subset C \quad , \quad \forall t \in [0,T] \quad .$$

PROPOSITION 2.1. The following statements are equivalent

(i) $\forall u_0 \in D(A) + G(0)$, $\forall f \in L^1(0,T;X)$

$(u_0 \in C$, $f(t) \in C$ a.e.t $\in [0,T]) \Rightarrow$

$\Rightarrow (\forall t \in [0,T]$, $u_\lambda(t,u_0,f) \in C)$

(ii) $\forall t \in [0,T]$, $\Phi(t) \in C$.

Proof of proposition 2.1.

The implication (ii) $\Rightarrow$ (i) follows easily from the representation of $u_\lambda(\cdot,u_0,f)$, (2.1) and (2.2). For the opposite implication, consider a sequence $\{u_0^n\} \to 0$ in X and $u_0^n \in C \cap D(A(0))$, $\forall n \in \mathbb{N}$. For instance we might take $u_0^n = J_{1/n}0$. We have

$$u_\lambda(t,u_0^n,0) = S_\lambda(t)u_0^n + \Phi(t) \in C \quad .$$

Since C is closed and $S_\lambda(t)$ is continuous in X, the proposition follows.

The following corollary, which is a direct consequence of Lemma 2.1, provides a sufficient condition on the family $A(t)$, to insure that $u_\lambda(t,u_0,f) \in C$ for all $t \in [0,T]$.

Corollary 2.1. Suppose that for all $t \in [0,T]$, $(I + \lambda A(t))^{-1}0 - e^{-t/\lambda}(I + \lambda A(0))^{-1}0 \in C$. Then for every $f \in L^1(0,T;X)$ with $f(t) \in C$ a.e.t $\in [0,T]$, and every $u_0 \in C \cap D(A(0))$, $u_\lambda(t,u_0,f) \in C$ for all $t \in [0,T]$. Moreover $u_\lambda(t,u_0,f) - e^{-t/\lambda}u_0 \in C$, for all $t \in [0,T]$.

Proof of the corollary.

From the definition of $A(t)$, it follows that $G(t)$ satisfies $G(t) + \lambda A(t)G(t) = 0$, in X for all $t \in [0,T]$. Therefore the assumptions of the corollary are equivalent to

$$G(t) - e^{-t/\lambda}G(0) \in C \quad .$$

By the invariance of J_λ and S_λ with respect to C we have that $\phi(t) \in C$ for all $t \in [0,T]$. From the representation of $u_\lambda(\cdot,u_0,f)$, we have

$$u_\lambda(t,u_0,f) - e^{-t/\lambda}u_0 = \sum_{n\geq 1} (\frac{1}{\lambda} J_\lambda)^n \frac{t^n}{n!} e^{-t/\lambda}u_0 + \phi(t) +$$

$$+ \int_0^t J_\lambda S_\lambda(t - s)f(s)ds \in C \quad .$$

This proves the corollary.

PROPOSITION 2.2. Let $f \in L^1(0,T;X)$, with $f(t) \in C$ a.e.t $\in [0,T]$ and assume that

$$G(t) + e^{-t/\lambda}(u_0 - G(0)) \in C \quad .$$

Then $u_\lambda(t,u_0,f) \in C$ for all $t \in [0,T]$.

Proof of Proposition 2.2.

Consider the representation of $u_\lambda(\cdot,u_0,f)$. We have

$$S_\lambda(t)u_0 + \phi(t) = \sum_{n\geq 1} (\frac{1}{\lambda} J_\lambda)^n \frac{t^n}{n!} e^{-t/\lambda}u_0 + [G(t) + e^{-t/\lambda}(u_0 - G(0))] +$$

$$+ \frac{1}{\lambda} J_\lambda \int_0^t S_\lambda(t - s)[G(s) + e^{-s/\lambda}(u_0 - G(0))]ds + S_\lambda(t) \int_0^t \frac{d}{ds} e^{-s/\lambda\, J_\lambda} u_0 ds =$$

$$= [G(t) + e^{-t/\lambda}(u_0 - G(0))] + \frac{1}{\lambda} J_\lambda \int_0^t S_\lambda(t - s)[G(s) + e^{-s/\lambda}(u_0 - G(0))]ds =$$

$$= H(t) \quad .$$

Therefore

$$u_\lambda(t,u_0,f) = H(t) + \int_0^t J_\lambda S_\lambda(t - s)f(s)ds \quad .$$

The assumptions imply that $H(t) \in C$ for all $t \in [0,T]$. Hence $u_\lambda(t,u_0,f) \in C$ for all $t \in [0,T]$.

B. Applications to boundary value problems

We consider here the case when $(PP)_t$ represents a partial differential equation with certain boundary conditions, and we give, in particular cases (including (E)), necessary and sufficient conditions on the boundary data, to insure a nonnegative solution.

Let Ω be a bounded domain in $\mathbb{R}^n$ which lies locally on one side of its smooth boundary Γ. $L^2(\Omega)$ is the space of square integrable real valued functions with respect to the Lebesgue measure over Ω and $H^1(\Omega)$ is the space of functions $\varphi \in L^2(\Omega)$ for which each of the (weak) partial derivatives $D_j = \frac{\partial\varphi}{\partial x_j}$ belongs to $L^2(\Omega)$, $1 \le j \le n$. If D_0 is the identity in $L^2(\Omega)$, the norm in $H^1(\Omega)$ can be expressed by

$$\|\varphi\|_{H^1(\Omega)} = \{ \sum_{j=0}^{n} \|D_j \varphi\|^2_{L^2(\Omega)} \}^{1/2} .$$

The spaces $H^k(\Omega)$, $k > 1$ integer can be defined analogously. If we denote by γ, the trace operator, restriction to Γ of elements in $H^1(\Omega)$, $H^{1/2}(\Gamma)$ is the range of γ and $H_0^1(\Omega)$ (dual $H^{-1}(\Omega)$) is the kernel of γ. We refer to [1] for more information about these Sobolev spaces.

For $u \in \mathcal{D}'(\Omega)$ define

$$Lu = - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + a_0(x)u$$

where $a_{ij} \in C^1(\bar{\Omega})$, b_i, $a_0 \in C(\bar{\Omega})$ and satisfy

$$a_0 \ge 0 \quad , \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad , \quad \nu > 0 \quad .$$

Here $|\xi|$ denotes the euclidean length of the vector $(\xi_1,\ldots,\xi_n)$ in $\mathbb{R}^n$.

If $\vec{\nu}(x) = (\nu_1(x),\ldots,\nu_n(x))$ denotes the outward unit normal to Γ, we denote with

$$\partial_L u = \sum_{i,j=1}^{n} a_{ij} u_{x_i} \nu_j(x) \qquad u \in H^2(\Omega)$$

the conormal derivative of u with respect to L.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \phi$ and consider the linear operator A defined by

$$D(A) \equiv \{u \in L^2(\Omega) : \; Lu \in L^2(\Omega) \;, \; \gamma_{\Gamma_1}(u) = 0 \;, \; \partial_L u|_{\Gamma_2} = 0\} \;.$$

$$Au = Lu \;.$$

If $a_0 \geq \alpha_0 > 0$ and α_0 is sufficiently large then L is coercive in $L^2(\Omega)$ and A is m-accretive (maximal monotone) in $L^2(\Omega)$. See [7, 17, 21] for details.

A satisfies the following "strong" maximum principle:

$$\forall u \in L^2(\Omega) \;, \; (u \geq 0 \;, \; u \not\equiv 0) \Rightarrow (J_\lambda u > 0 \text{ in } \Omega) \;.$$

In particular the closed convex cone C of the nonnegative functions in $L^2(\Omega)$ is invariant under J_λ.

(α) The case of Dirichlet boundary data.

Here we suppose $\Gamma_2 = \phi$. Let $g \in C([0,T];H^{3/2}(\Gamma))$ and consider the family of operators $\{A(t)\}_{t\in[0,T]}$ defined as follows:

$$D[A(t)] \equiv \{u \in L^2(\Omega) : \; Lu \in L^2(\Omega) \text{ and } \gamma(u(t)) = g(t) \text{ a.e. } t \in [0,T]\}$$

$$A(t)u = Lu \;.$$

Let $G(t) \in C([0,T];L^2(\Omega))$, $G(t) \in D[A(t)]$ $\forall t \in [0,T]$ be the unique solution of

$$\text{(2.3)} \qquad \begin{cases} G(t) + \lambda LG(t) = 0 \\ \gamma(G(t)) = g(t) \text{ on } \Gamma \;. \end{cases}$$

Then the operators $A(t)$ are of the form described in part A, that is

$$D[A(t)] = D(A) + G(t) \text{ and}$$

$$\forall u \in D[A(t)] \;, \; A(t)u = A(u - G(t)) - \frac{1}{\lambda} G(t) \;.$$

Therefore for all $u_0 \in D(A) + G(0)$ and $f \in L^1(0,T;L^2(\Omega))$ there exists a unique solution of

$$\begin{cases} u \in C([0,T];L^2(\Omega)) \quad , \quad u + \lambda Lu \in W^{1,1}(0,T;L^2(\Omega)) \\ \frac{\partial}{\partial t}(u + \lambda Lu) + Lu = f \quad , \quad u(0) = u_0 \\ \gamma(u) = g \quad \text{on} \quad \Gamma \quad . \end{cases}$$

Moreover the solution, which will be denoted by $u_\lambda(\cdot,u_0,f)$ can be represented as in Lemma 2.1, namely

$$\begin{aligned} u_\lambda(t,u_0,f) &= S_\lambda(t)u_0 + [G(t) - e^{-t/\lambda}G(0)] + \\ &+ \frac{1}{\lambda}\int_0^t J_\lambda S_\lambda(t-s)[G(s) - e^{-s/\lambda}G(0)]ds + \int_0^t J_\lambda S_\lambda(t-s)f(s)ds \quad . \end{aligned} \tag{2.4}$$

Remark 2.2. We will comment later on the meaning of the representation (2.4).

The results of the previous part A carry over to the present situation if we choose C to be the closed convex cone of the nonnegative functions in $L^2(\Omega)$. In fact we can obtain more precise results.

Theorem 2.1. The following statements are equivalent:

(i) $\forall u_0 \in D(A) + G(0) \quad , \quad \forall f \in L^1(0,T;L^2(\Omega))$

$(u_0 \geq 0 \quad , \quad f(t) \geq 0 \ \text{a.e.} \ t \in [0,T]) \Rightarrow (u_\lambda(t,u_0,f) \geq 0 \quad , \quad \forall t \in [0,T])$.

(ii) $\forall t \in [0,T] \quad g(t) \geq e^{-t/\lambda}g(0)$.

Proof of Theorem 2.1.

In view of (2.3), $G(t) - e^{-t/\lambda}G(0)$ satisfies

$$[G(t) - e^{-t/\lambda}G(0)] + \lambda L[G(t) - e^{-t/\lambda}G(0)] = 0$$

$$\gamma[G(t) - e^{-t/\lambda}G(0)] = g(t) - e^{-t/\lambda}g(0) \geq 0 \quad .$$

Therefore by the maximum principle (ii) implies $G(t) - e^{-t/\lambda}G(0) \geq 0$. Hence the implication (ii) $\Rightarrow$ (i) follows from (2.4). Now let (i) hold and consider the representation (2.4). By Proposition 2.1, $\Phi(t) \geq 0$ $\forall t \in [0,T]$ and $u_\lambda(t,u_0,f) \geq$ $\geq S_\lambda(t)u_0$, $\forall t \in [0,T]$. It follows that

(2.5) $$u_\lambda(t,u_0,f) - e^{-t/\lambda}u_0 \geq \sum_{n\geq 1} (\tfrac{1}{\lambda} J_\lambda)^n \frac{t^n}{n!} e^{-t/\lambda}u_0 \geq 0 \quad .$$

Hence

$$\gamma(u_\lambda(t,u_0,f) - e^{-t/\lambda}u_0) = g(t) - e^{-t/\lambda}g(0) \geq 0 \quad .$$

The next proposition supplies a sufficient condition on the data u_0 and $g(\cdot)$ on the whole parabolic boundary of $\Omega \times [0,T]$, to insure that $u_\lambda(t,u_0,f) \geq 0$.

<u>PROPOSITION 2.3</u>. Let $f \in L^1[0,T;L^2(\Omega)]$, with $f(t) \geq 0$ for a.e. $t \in [0,T]$, and assume that

$$G(t) + e^{-t/\lambda}(u_0 - G(0)) \geq 0 \qquad \forall t \in [0,T] \quad .$$

Then $u_\lambda(t,u_0,f) \geq 0$, $\forall t \in [0,T]$.

<u>Proof of Proposition 2.3</u>.

This is the content of proposition 2.2.

<u>Remarks 2.3</u>. (a) Proposition 2.3 contains as particular cases two different kinds of results:

(i) If $G(t) \geq e^{-t/\lambda}G(0)$, then $u_\lambda(t,u_0,f) \geq 0$ for any $u_0 \geq 0$ (Theorem 2.1) .

(ii) If $u_0 \geq G(0)$, then $u_\lambda(t,u_0,f) \geq 0$ for any $g(t) \geq 0$.

The latter case was observed in [20].

(b) Since $u_0 \in D[A(0)]$, $\gamma(u_0) = \gamma(G(0))$, hence the assumptions of the proposition imply that we must have $G(t) \geq 0$.

<u>PROPOSITION 2.4 (Strong maximum principle)</u>. Let $f \in L^1[0,T;L^2(\Omega)]$ with $f(t) \geq 0$ a.e. $t \in [0,T]$, $u_0 \geq 0$ and assume that $g(t) \geq e^{-t/\lambda}g(0)$. Then if either

u_0 or f is not identically zero, we have

$$u_\lambda(t,u_0,f) \quad e^{-t/\lambda}u_0 \qquad \forall t \quad [0,T] \quad .$$

Proof of Proposition 2.4.

If $u_0 \not\equiv 0$, $J_\lambda u_0 \quad 0$. Hence in this case the proposition follows from (2.5). If $f \not\equiv 0$ the proof is similar, starting from the representation (2.4).

Remark 2.4. This result was observed by Ting [23] for homogeneous boundary data and for $u_0 \quad 0$ in . It also answers a question raised in [20] on the possibility of a strong maximum principle for pseudoparabolic equations with nonhomogeneous Dirichlet data.

We comment briefly on the representation (2.4). For simplicity we assume $f(t) \quad 0$.

Setting

$$U^0 = S_\lambda(t)u_0 \quad ,$$

it is easy to verify that U^0 is a solution of

$$(P_1) \qquad \begin{cases} \dfrac{d}{dt}(U^0 + \lambda LU^0) + LU^0 = 0 \\[2ex] \gamma(U^0(t)) = e^{-t/\lambda}g(0) \\[2ex] U^0(0) = u_0 \end{cases}$$

whereas $\Phi(t)$ satisfies

$$(P_2) \qquad \begin{cases} \dfrac{d}{dt}(\Phi(t) + \lambda L\Phi(t)) + L\Phi(t) = 0 \\[2ex] \lambda(\Phi(t)) = g(t) - e^{-t/\lambda}g(0) \\[2ex] \Phi(0) = 0 \quad . \end{cases}$$

Therefore the solution $u_\lambda(\cdot,u_0,0)$ can be separated into the solution U^0 of the pseudoparabolic problem (P_1), and $\Phi(\cdot)$ solution of a pseudoparabolic problem with homogeneous initial data (P_2).

In order to single out some features of this kind of equation, we consider a few limiting cases. Let $\{u_0^n\}$ be a sequence converging to zero in $L^2(\Omega)$, $u_0^n \in D[A(0)]$, $\forall n \in \mathbb{N}$ and let $u_\lambda(t,u_0^n,0)$ be the corresponding solutions given by (2.4).

I. $g(t) \equiv 1$.

In this case we have

$$u_\lambda(t,u_0^n,0) = S_\lambda(t)u_0^n + (1 - e^{-t/\lambda}) + \frac{1}{\lambda}\int_0^t J_\lambda S_\lambda(s-t)(1 - e^{-s/\lambda})ds$$

and for all $n \in \mathbb{N}$, $\gamma(u_\lambda(t,u_0^n,0)) = 1$. As $n \to \infty$, $u_\lambda(t,u_0^n,0)$ converges to the "generalized" solution of

$$\frac{d}{dt}(u + \lambda Lu) + Lu = 0$$

$$\gamma(u) = 1 \quad \text{on } \Gamma \quad , \quad u(0) = 0 \quad ,$$

which satisfies $\gamma(u(t)) = 1 - e^{-t/\lambda}$ on Γ.

Therefore the limit solution does not follow any more the boundary data. This fact has been observed in [11].

II. $g_1(t) = e^{-t/\lambda}g_1(0) \quad , \quad g_1(0) > 0$.

In this case $G_1(t) = e^{-t/\lambda}G_1(0)$ and by (2.4)

$$u_\lambda^1(t,u_0^n,0) = S_\lambda(t)u_0^n$$

$$\gamma(u_\lambda^1(t,u_0^n,0)) = e^{-t/\lambda}g_1(0) > 0 \quad .$$

As $n \to \infty$, $u_\lambda^1(\cdot,u_0^n,0) \to 0$ in $L^2(\Omega)$. Therefore the "generalized" solution of

$$\frac{d}{dt}(u^1 + \lambda Lu^1) + Lu^1 = 0$$

$$\gamma(u^1(t)) = e^{-t/\lambda}g_1(0)$$

$$u^1(0) \equiv 0$$

is the identically zero function, in spite of the fact that the boundary data are positive.

III. $g_2(0) = 0$, $g_2(t) > 0$ for $t > 0$.

Since $u_0 = 0 \in D[A(0)]$ the problem

$$u \in C([0,T];L^2(\Omega));u + \lambda Lu \in W^{1,1}(0,T;L^2(\Omega))$$

$$\frac{\partial}{\partial t}(u + \lambda Lu) + Lu = 0$$

$$\gamma(u(t)) = g_2(t) > 0 \qquad t > 0$$

$$u(0) = 0$$

admits a strong solution $u_\lambda^2(\cdot,0,0)$ given by

$$u_\lambda^2(t,0,0) = \phi(t) = G_2(t) + \frac{1}{\lambda}\int_0^t J_\lambda S_\lambda(t-s)G_2(s)ds \quad .$$

Since $g_2(t) > 0$ for $t > 0$, $\phi(t) > 0$ and hence

$$u_\lambda^2(t,0,0) > 0 \qquad \forall t > 0 \quad .$$

Consider now a datum $0 < g_2(t) < e^{-t/\lambda}g_1(0)$ where $g_1(0)$ is the datum in case II. Then the previous remarks show that

$$u_\lambda^2(t,0,0) > u_\lambda^1(t,0,0) \equiv 0 \quad ,$$

in spite of the fact that the boundary data satisfy the opposite inequality.

These facts are in striking contrast with the behaviour of the solutions of the classical heat equation.

(β) The case of Neumann boundary data.

Here we assume $\Gamma_1 = \phi$. Let $p \in C([0,T];H^{1/2}(\Gamma))$ and consider the family $\{A(t)\}_{t\in[0,T]}$ defined as follows:

$$D[A(t)] = \{u \in L^2(\Omega) : \partial_L u = p(t) \ , \ Lu \in L^2(\Omega)\}$$

$$A(t)u = Lu \ .$$

(see [17] for details)

Let $G(\cdot) \in C([0,T];L^2(\Omega))$, $G(t) \in D[A(t)]$ $\forall t \in [0,T]$ be the unique solution of

$$G(t) + \lambda LG(t) = 0$$

$$\partial_L G(t) = p(t) \quad \text{on} \quad \Gamma \ , \quad \forall t \in [0,T] \ .$$

Then the operators $A(t)$ are of the form described in part A, that is

$$D[A(t)] = D(A) + G(t)$$

$$\forall u \in D[A(t)] \ , \quad A(t)u = A(u - G(t)) - \frac{1}{\lambda} G(t) \ .$$

Therefore for all $u_0 \in D(A) + G(0)$ and $f \in L^1(0,T;L^2(\Omega))$ there exists a unique solution of

$$\begin{cases} u \in C([0,T];L^2(\Omega)) \ , \quad u + \lambda Lu \in W^{1,1}(0,T;L^2(\Omega)) \\ \frac{\partial}{\partial t}(u + \lambda Lu) + Lu = f \ , \quad u(0) = u_0 \\ \partial_L u = g \quad \text{a.e. on} \quad \Gamma \ . \end{cases}$$

Moreover the solution, which will be denoted by $u_\lambda(t,u_0,f)$, can be represented as in (2.4) with the obvious changes in the meaning of the symbols.

<u>PROPOSITION 2.5</u>. Let $f \in L^1(0,T;L^2(\Omega))$ with $f(t) \geq 0$ a.e. $t \in [0,T]$, $u_0 \in D[A(0)]$, $u_0 \geq 0$ and suppose that $p(t) \geq e^{-t/\lambda} p(0)$ a.e. on Γ and $\forall t \in [0,T]$. Then

$$u_\lambda(t,u_0,f) \geq e^{-t/\lambda} u_0 \qquad \forall t \in [0,T] \ .$$

Moreover if either u_0 or f is not identically zero, then

$$u_\lambda(t,u_0,f) > e^{-t/\lambda} u_0 \ .$$

Proof of Proposition 2.5.

For all $t \in [0,T]$, $G(t) - e^{-t/\lambda}G(0)$ satisfies

$$(2.6)\qquad \begin{aligned} &[G(t) - e^{-t/\lambda}G(0)] + \lambda L[G(t) - e^{-t/\lambda}G(0)] = 0 \\ &\partial_L[G(t) - e^{-t/\lambda}G(0)] = p(t) - e^{-t/\lambda}p(0) \geq 0 \quad . \end{aligned}$$

Therefore by the maximum principle, $G(t) - e^{-t/\lambda}G(0) \geq 0 \ \ \forall t \in [0,T]$, and from (2.4)

$$u_\lambda(t,u_0,f) - e^{-t/\lambda}u_0 \geq \sum_{n\geq 1} (\tfrac{1}{\lambda} J_\lambda)^n \frac{t^n}{n!} e^{-t/\lambda}u_0 +$$

$$+ \int_0^t J_\lambda S_\lambda(t - s)f(s)ds \geq 0 \quad .$$

The second statement is obvious.

Remark 2.5. The assumptions in Proposition 2.5 do not impose any signum restriction on $p(t)$. In particular $p(t)$ could be negative, as long as $p(t) \geq e^{-t/\lambda}p(0)$.

Remark 2.6. The condition $p(t) \geq e^{-t/\lambda}p(0)$ is not necessary. In fact the following weaker assumption on $p(t)$ is sufficient in order that $(f(t) \geq 0$, $u_0 \geq 0) \Rightarrow (u_\lambda(t,u_0,f) \geq 0)$.

For all $t \in [0,T]$ let $\varphi_h(t)$ be the solution of

$$(2.7)\qquad \begin{cases} \varphi_h(t) + \lambda L\varphi_h(t) = 0 \\ \partial_L\varphi_h(t) = h(t) \geq 0 \quad , \quad h \in L^2(\Gamma)^+ \quad . \end{cases}$$

Then $G(t) - e^{-t/\lambda}G(0) \geq 0$ [and hence $(f \geq 0,\ u_0 \geq 0) \Rightarrow u_\lambda(t,u_0,f) \geq 0)]$ if

$$(2.8)\qquad \begin{cases} \int_\Gamma [p(t) - e^{-t/\lambda}p(0)]\gamma(\varphi_h(t))d\sigma \geq 0 \\ \text{for all } h \geq 0 \quad , \quad h \in L^2(\Omega)^+ \quad . \end{cases}$$

Indeed by multiplying (2.6) by $\varphi_h(t)$ and integrating by parts we are led to the identity

$$\int_\Gamma \gamma[G(t) - e^{-t/\lambda}G(0)]h\, d\gamma = \int_\Gamma [p(t) - e^{-t/\lambda}p(0)]\gamma[\varphi_h(t)]d\gamma$$

Therefore if (2.8) holds, from the arbitrarity of $h \in L^2(\Gamma)^+$ we deduce

$$\gamma[G(t) - e^{-t/\lambda}G(0)] \geq 0 \qquad \forall t \in [0,T] \quad .$$

This together with (2.6) implies $G(t) - e^{-t/\lambda}G(0) \geq 0$.

<u>Remark 2.7</u>. The condition (2.8) is in fact weaker than $p(t) - e^{-t/\lambda}p(0) \geq 0$. This is shown by the following counterexample.

Consider $L = -\frac{\partial^2}{\partial x^2}$ in $(0,1)$, $\lambda = 1$. Then all the solutions of (2.7) are given by

(2.9) $$\varphi_{A,B} = Bchx - Ashx \quad , \quad A \geq 0 \quad , \quad Btgh1 \geq A \quad .$$

The function

$$p = \begin{cases} -\frac{1}{3} \quad , & x = 0 \\ \frac{1}{ch1} \quad , & x = 1 \end{cases}$$ on $\partial[0,1]$, satisfies

$$\int_{\partial[0,1]} p\, \varphi_{A,B} \geq 0 \quad \text{for all } \varphi_{A,B} \text{ given by (2.9)} \quad .$$

The above can be restated by saying that the cone $\{\gamma(\varphi_h) \; ; \; h \in L^2(\Gamma)^+\}$ is <u>not</u> dense in $L^2(\Gamma)^+$.

<u>Remark 2.8</u>. It is not difficult to show that a necessary condition is the following:

$$[(f \geq 0 \quad , \quad u_0 \geq 0) \Rightarrow u_\lambda(t,u_0,f) \geq 0] \Rightarrow \int_0^t \int_\Gamma [p(\tau) - e^{-\tau/\lambda}p(0)]d\sigma d\tau \geq 0 \quad .$$

<u>Remark 2.9</u>. Results similar to the ones in Theorem 2.1 and Propositions 2.3-2.5, can be obtained for the case of mixed boundary conditions.

(γ) $L^\infty(\Omega)$-estimates.

We briefly indicate how the previous results can be used to obtain a priori estimates on the solution.

For simplicity we take $A = -\Delta$ and $f = 0$, and consider the problem

$$(2.10)\qquad \begin{cases} \frac{d}{dt}[u(t) - \lambda\Delta u(t)] - \Delta u(t) = 0 \quad \text{in} \quad L^2(\Omega) \\ \gamma(u(\cdot)) = g(\cdot) \in C([0,T] : H^{3/2}(\Gamma)) \\ u(0) = u_0 \quad , \quad \gamma(u_0) = g(0) \quad . \end{cases}$$

Set

$$\sup_{t\in[0,T]} \left\| \frac{g(t) - e^{-t/\lambda}g(0)}{1 - e^{-t/\lambda}} \right\|_{L^\infty(\Gamma)} = M(T) \quad .$$

PROPOSITION 2.6. The solution u of (2.10) satisfies the estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le \max\{M(T) \quad , \quad \|u_0\|_{L^\infty(\Omega)}\} \quad , \quad \forall t \in [0,T] \quad .$$

Remark 2.10. The quotient

$$\frac{g(t) - e^{-t/\lambda}g(0)}{1 - e^{-t/\lambda}} = g(t) + \frac{e^{-t/\lambda}}{1 - e^{-t/\lambda}}(g(t) - g(0))$$

converges to $g(t)$, $t \in (0,T]$ as $\lambda \to 0$. Hence for $\lambda = 0$, as a limit case we find the known estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le \max\{ \sup_{t\in[0,T]} \|g(t)\|_{L^\infty(\Gamma)} \quad , \quad \|u_0\|_{L^\infty(\Omega)}\}$$

for the solution u of the classical heat equation.

Proof of Proposition 2.6.

Set $k = \max\{M(T), \|u_0\|_{L^\infty(\Omega)}\}$. If $k = \infty$ then the statement is vacuous. Let then $k < \infty$, and set $v = k - u$. It is immediate to verify that v solves:

$$\frac{d}{dt}(v - \lambda\Delta v) - \Delta v = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$\gamma(v) = k - g$$

$$v(0) = k - u_0 \geq 0 \quad , \quad \gamma(v_0) = k - g(0) \quad .$$

For the particular choice of k, $\gamma(v)$ satisfies

$$\gamma(v(t)) \geq e^{-t/\lambda}\gamma(v(0)) \quad .$$

Hence by Theorem 2.1, $u(t) \leq k$ in $\Omega \times [0,T]$. The bound from below is derived analogously.

An estimate of the same nature can be derived on the gradient of u. Suppose u satisfies the first of (2.10) and assume that we are given the functions $\frac{\partial u}{\partial x_i}\Big|_\Gamma = p_i(t) \in C([0,T];H^{1/2}(\Gamma))$. Set

$$M(T) = \sup_{\substack{t\in[0,T] \\ 1\leq i\leq n}} \left\| \frac{p_i(t) - e^{-t/\lambda}p_i(0)}{1 - e^{-t/\lambda}} \right\|_{L^\infty(\Gamma)} \quad .$$

Then u_{x_i} $1 \leq i \leq n$ are solutions of

$$\frac{\partial}{\partial t}(u_{x_i} - \lambda\Delta u_{x_i}) - \Delta u_{x_i} = 0 \quad \text{in} \quad \Omega \times [0,T]$$

$$\gamma(u_{x_i}(t)) = p_i(t) \qquad \forall t \in [0,T]$$

$$u_{x_i}(0) = u_{0_{x_i}} \quad .$$

From Proposition 2.6 we easily deduce the estimate

$$\|\nabla_x u(t)\|_{L^\infty(\Omega)} \leq \max\{M(T) \ ; \ \|\nabla u_0\|_{L^\infty(\Omega)}\} \quad .$$

Estimates of the same kind can be obtained for more general operators A and $f \neq 0$. However they fall beyond the scope of this work.

2. CASE WHEN (PP) IS GOVERNED BY NONLINEAR OPERATORS

Here we study the properties of maximum principle for:

$$\text{(PP)} \qquad \frac{d}{dt}(u + Au) + Au \ni f \ , \quad u(0) = u_0 \ , \quad Au(0) = w_0 \ ,$$

when A is a nonlinear perturbation of $-\Delta$. Let β be a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$ with [illegible] (see [7]) and [illegible] a bounded open set in $\mathbb{R}^N$. We successively consider:

(a) $A = $ "$-\Delta\beta$" in $L^1(\Omega)$, that is:

$$A = \{[u,w] \in L^1(\Omega) \times L^1(\Omega) \ ; \ \exists h \in W_0^{1,1}(\Omega) \text{ with } w = -\Delta h \ , \ h(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega\} \ .$$

(b) [illegible]

$$A = \{[u,w] \in L^2(\Omega) \times L^2(\Omega) \ ; \ u \in H^2(\Omega) \ , \ -\frac{\partial u}{\partial n} \in \beta(u) \text{ a.e. on } \partial\Omega \ , \ w = -\Delta u\} \ .$$

(c) $A = $ "$-\Delta + \beta$" in $L^2(\Omega)$, that is:

$$A = \{[u,w] \in H_0^1(\Omega) \times L^2(\Omega) \ ; \ \exists h \in L^2(\Omega) \text{ with } -\Delta u + h = w \ , \ h(x) \in \beta(u(x)) \text{ a.e. } x\} \ .$$

These operators are m-accretive in $L^1(\Omega)$, $L^2(\Omega)$, $L^2(\Omega)$ respectively (see [illegible], [illegible], [5]) and satisfy the maximum principle in various forms (see [9], [5]); in particular, if u is the solution of the associated parabolic equation:

$$\text{(P)} \qquad \frac{du}{dt} + Au \ni f \ , \quad u(0) = u_0 \ ,$$

then: $(u_0 \geq 0, \ f \geq 0) \Rightarrow (u(t) \geq 0)$. This is not generally the case for the solutions of (PP): here, we give in each of the cases (a), (b), (c) a necessary and sufficient condition on β in order for (PP) to have such a property.

A) Case (a): $A = $ "$-\Delta\beta$"

This case is a "good" one in the sense that (PP) satisfies properties of maximum principle without any extra assumption on β.

We denote by A the m-accretive operator in $L^1(\Omega)$ defined in (a). By the results of the section 1, for $f \in L^1(0,T;L^1(\Omega))$, $[u_0,w_0] \in A$ and $\lambda > 0$, there exists a unique solution of:

$$(E_1) \qquad \frac{d}{dt}(u - \lambda\Delta\beta u) - \Delta\beta u = f \quad , \quad u(0) = u_0 \quad , \quad -\Delta\beta u(0) = w_0 \quad .$$

We denote it by $u_\lambda(\cdot,u_0,w_0,f)$.

Remark 3.1. The equation (E_1) represents a model of diffusion in fractured porous media and also a nonlinear model for a two temperatures theory of heat conduction (see Showalter [18]).

PROPOSITION 3.1. The following implications hold:

(i) $(u_0 \geq 0 \quad , \quad f \geq 0) \Rightarrow (\forall t \in [0,T] \quad , \quad \forall w_0 \in Au_0 \quad , \quad u_\lambda(t,u_0,w_0,f) \geq 0)$

(ii) $(|u_0| \leq k) \Rightarrow (\forall t \in [0,T] \quad , \quad \forall w_0 \in Au_0 \quad , \quad |u_\lambda(t,u_0,w_0,0)| \leq k)$.

Remark 3.2. We note that the results above carry over to more general situations. Namely they remain valid for equations such as

$$\frac{\partial}{\partial t}(u - \lambda\Delta\beta u) - \Delta\gamma u \ni f$$

where γ is another maximal monotone graph with $0 \in \gamma(0)$. Moreover $-\Delta$ could be replaced by any linear operator L in $L^1(\Omega)$ satisfying the following "maximum principles" (Here sgn^+ denotes the maximal monotone operator defined by:

$$\operatorname{sgn}^+ r = \begin{cases} 0 & \text{if } r < 0 \\ [0,1] & \text{if } r = 0 \\ 1 & \text{if } r > 0 \end{cases}) \quad .$$

$$(M_0) \begin{cases} \forall u \in D(L) \quad , \quad \forall w \in L^\infty(\Omega) \quad \text{with} \quad w(x) \in \operatorname{sgn}^+ u(x) \quad \text{a.e.} \quad x \in \Omega \\ \int_\Omega wLu \geq 0 \quad . \end{cases}$$

$$(M) \begin{cases} \forall k \geq 0 \quad , \quad \forall u \in D(L) \quad , \quad \forall w \in L^\infty(\Omega) \quad \text{with} \quad w(x) \in \operatorname{sgn}^+(u(x) - k) \quad \text{a.e.} \quad x \in \Omega \\ \int_\Omega wLu \geq 0 \quad . \end{cases}$$

In [9], it is shown that (M) is satisfied for operators such as:

$$Lu = -\sum_{i,j}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} (b_i(x)u)_{x_i} + a_0 u \quad ,$$

with $D(L) = \{u \in W_0^{1,1}(\Omega) \ ; \ Lu \in L^1(\Omega)\}$ and:

$$a_{ij} \ , \ b_i \in C^1(\bar{\Omega}) \ , \ a_0 \in L^\infty(\Omega) \ , \ a_0 \geq 0 \ , \ a_0 + \sum_i (b_i)_{x_i} \geq 0 \ \text{ a.e.}$$

Maximum principles of type (M) for nonlinear operators have been extensively studied in [5] and [6].

The proposition is a particular case of the following results which holds for any measured space Ω.

<u>Theorem 3.1</u>. Let L be a <u>linear</u> operator on $L^1(\Omega)$, β, γ two maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ containing the origin and $f \in L^1(0,T;L^1(\Omega))$. Let $u \in C([0,T];L^1(\Omega))$ satisfying:

$$\begin{cases} \exists h \ , \ \hat{h} \in L^1(0,T;L^1(\Omega)) \ \text{ with } \ h(t) \in \beta(u(t)) \ , \ \hat{h}(t) \in \gamma(u(t)) \ \text{ a.e. } t \\ h(t) \ , \ \hat{h}(t) \in D(L) \ \text{ a.e. } t \ , \ u + \lambda Lh \in W^{1,1}(0,T;L^1(\Omega)) \\ \dfrac{d}{dt}(u + \lambda Lh) + L\hat{h} = f \ . \end{cases}$$

Then:

(i) **If** L **satisfies** (M_0):

$$(u(0) \geq 0 \ , \ f \geq 0) \Rightarrow (\forall t \in [0,T] \ , \ u(t) \geq 0) \ .$$

(ii) **If** L **satisfies** (M) **and** $f \equiv 0$:

$$(|u(0)| \leq k) \Rightarrow (\forall t \in [0,T] \ , \ |u(t)| \leq k) \ .$$

<u>**Proof of Theorem 3.1.**</u>

Formally, the idea of the proof of (i) is to multiply the equation by $\operatorname{sgn}^-(\lambda h_t + \hat{h}) = \text{"}\operatorname{sgn}^-(\lambda\beta'(u)u_t + \gamma(u))\text{"}$, where $\operatorname{sgn}^- r = \operatorname{sgn}^+(-r)$.

We will do this in an approximated way. For $\varepsilon > 0$, set

$$(3.1) \qquad \eta(t,\varepsilon) = \frac{u(t) - u(t-\varepsilon)}{\varepsilon} + \lambda L \frac{h(t) - h(t-\varepsilon)}{\varepsilon} + L\hat{h}(t) - f(t) \ .$$

As $u + Lh \in W^{1,1}(0,T;L^1(\Omega))$, (3.1) and the equation yield:

$$\lim_{\varepsilon \to 0^+} \int_0^T \int_\Omega |n(t,\varepsilon)| = 0 \quad . \tag{3.2}$$

For t and ε fixed we consider $w \in L^\infty(\Omega)$ defined by:

$$w = \begin{cases} 1 & \text{on } [h(t) > 0] \cup ([h(t) = 0] \cap [u(t) > 0]) \\ 0 & \text{on } [h(t) < 0] \cup ([h(t) = 0] \cap [u(t) \le 0]) \end{cases} \quad .$$

Then $w \in \operatorname{sgn}^-(h(t))$ and since $h(t) \in \beta(u(t))$ a.e. on Ω

$$w \cdot u(t) = -u^-(t) \quad \text{a.e. on } \Omega \quad . \tag{3.3}$$

Next let $\hat{w} \in L^\infty(\Omega)$ with $\hat{w} \in \operatorname{sgn}^-\left(\frac{h(t) - h(t-\varepsilon)}{\varepsilon} + n(t)\right)$ such that $\hat{w} = w$ on $[h(t) = h(t-\varepsilon)]$. Since $h(t) \in \beta(u(t))$, $h(t-\varepsilon) \in \beta(u(t-\varepsilon))$, the monotonicity of sgn^- gives:

$$\frac{u(t) - u(t-\varepsilon)}{\varepsilon}\, \hat{w} \le \frac{u(t) - u(t-\varepsilon)}{\varepsilon}\, w \quad \text{a.e. on } \Omega \quad . \tag{3.4}$$

Multiplying (3.1) by $\hat{w}$, integrating over Ω and taking in account that $f \ge 0$ and the condition M_0, we obtain:

$$-\int_\Omega |n(t,\varepsilon)| \le \int_\Omega \frac{u(t) - u(t-\varepsilon)}{\varepsilon}\, \hat{w} \quad . \tag{3.5}$$

The inequalities (3.5), (3.4) and (3.3) imply:

$$-\int_\Omega |n(t,\varepsilon)| \le \int_\Omega \frac{1}{\varepsilon}(-u^-(t) - wu(t-\varepsilon)) \le \frac{1}{\varepsilon}\left[\int_\Omega u^-(t-\varepsilon) - \int_\Omega u^-(t)\right] \quad .$$

Letting $\varepsilon \to 0$, by (3.2) we have:

$$\frac{d}{dt}\int_\Omega u^-(t) \le 0 \quad \text{in } \mathcal{D}'(]0,T[) \quad .$$

This proves (i).

The proof of (ii) is exactly the same. We multiply (3.1) by a suitable selection $\hat{w}$ out of $\operatorname{sgn}^+[\lambda \frac{h(t) - h(t - \varepsilon)}{\varepsilon} + \hat{h}(t) - \theta)$ where $\theta \in \beta(k)$ and $k \geq 0$ is to be selected. First we choose $w \in \operatorname{sgn}^+(\hat{h}(t) - \theta)$ such that $(u(t) - k)w = (u(t) - k)^+$; then we select $\hat{w}$ with the requirement that it agrees with w on the set $[h(t) = h(t - \varepsilon)]$. Then, the same computation as above gives (this time we use (M)):

$$\int_\Omega \frac{u(t) - u(t - \varepsilon)}{\varepsilon} w \leq \int_\Omega \frac{u(t) - u(t - \varepsilon)}{\varepsilon} \hat{w} \leq \int_\Omega |\eta(t,\varepsilon)| \quad .$$

This implies

$$\frac{1}{\varepsilon} [\int_\Omega (u(t) - k)^+ - (u(t - \varepsilon) - k)^+] \leq \int_\Omega |\eta(t,\varepsilon)| \quad .$$

When ε goes to 0, by (1) we obtain:

$$\frac{d}{dt} \int_\Omega (u(t) - k)^+ \leq 0 \quad \text{in} \quad \mathcal{D}'(]0,T[) \quad .$$

Choosing $k = \|u(0)\|_\infty$ in the above gives a bound from above for $u(t)$. The bound from below is found analogously.

B) <u>Case (b): "A = "$-\Delta$" with nonlinear boundary conditions</u>

Here Ω is a bounded open set with a smooth boundary Γ and β a maximal monotone operator on $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta 0$ (β^0 denotes its minimal section). Then the operator:

$$A = \{[u,w] \in L^2(\Omega) \times L^2(\Omega) \; ; \; u \in H^2(\Omega) \; , \; -\Delta u = w \; , \; -\frac{\partial u}{\partial n} \in \beta(u) \quad \text{a.e. on } \Gamma\} \quad ,$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ, is m-accretive (or maximal monotone) in $L^2(\Omega)$ (see [8]). By the existence results of the section 1, for any $u_0 \in D(A)$ and $f \in L^1(0,T;L^2(\Omega))$ there exists a unique solution of

$$(E_2) \qquad \begin{cases} \frac{\partial}{\partial t}(u - \lambda\Delta u) - \Delta u = f \quad , \quad u(0) = u_0 \quad , \quad (\lambda > 0) \\ -\frac{\partial u}{\partial n} \in \beta(u) \quad \text{a.e. on } \Gamma \quad . \end{cases}$$

We denote it by $u_\lambda(t,u_0,f)$.

Theorem 3.2. The following statements are equivalent:

(i) $\forall u_0 \in D(A)$

$$(u_0 \geq 0) \Rightarrow (\forall t \in [0,T] \quad , \quad u_\lambda(t,u_0,0) \geq 0) \quad .$$

(ii) The application $[r \in]0,\infty[\cap D(\beta) \to \frac{\beta^0(r)}{r}]$ is nondecreasing

(iii) $\forall u_0 \in D(A) \quad , \quad \forall f \in L^1(0,T;L^2(\Omega))$

$$(u_0 \geq 0 \quad , \quad f \geq 0) \Rightarrow (\forall t \in [0,T] \quad , \quad u_\lambda(t,u_0,f) \geq 0) \quad .$$

Moreover, if (ii) is satisfied:

(iv) $(u_0 \geq 0 \quad , \quad f \geq 0 \;\text{ and }\; u_0 \not\equiv 0 \;\text{ or }\; f \not\equiv 0) \Rightarrow$

$$\Rightarrow (u_\lambda(t,u_0,f) > e^{-t/\lambda}u_0 \quad \forall t \in]0,T]) \quad .$$

Remark 3.3. It is rather surprising that (ii) is the necessary and sufficient condition for the property of maximum principle (i). Even more surprising is that it is independent of λ. Note that:

$$(\beta^0 \;\text{ convex on }\; [0,\infty] \cap D(\beta)) \Rightarrow \text{(ii)} \quad , \quad \text{and}$$

$$\text{(ii)} \Rightarrow (\forall r,s \in [0,\infty[\quad , \quad \beta^0(r+s) \geq \beta^0(r) + \beta^0(s)) \quad .$$

Moreover, it is easy to verify that β satisfies (ii) if and only if its Yosida approximations do. This is equivalent to:

$$\forall \mu > 0 \quad , \quad \text{a.e.} \quad \beta_\mu(r) \leq r\beta'_\mu(r) \quad .$$

Of interest is also the following proposition which supplies a condition on u_0 (independent of β) that insures the nonnegativity of $u_\lambda(\cdot,u_0,f)$.

PROPOSITION 3.2. Let $u_0 \in D(A)$, $u_0 \geq 0$ and let G be the solution of

$$G - \lambda\Delta G = 0 \quad , \quad G = u_0 \;\text{ a.e. on }\; \partial\Omega \quad .$$

Then:

$$(u_0 \geq G) \Rightarrow (\forall f \geq 0 \quad , \quad \forall t \in [0,T] \quad , \quad u_\lambda(t,u_0,f) \geq 0) \quad .$$

Remark 3.4. The condition above is similar to the one in [20] or in the proposition 2.3 here for the case of Dirichlet boundary data.

In particular, if $u_0 = 0$ on Γ and $u_0 \geq 0$, then $u_\lambda(t,u_0,0) \geq 0$ without any extra assumptions on β.

The main tool in the proof of (i) $\Rightarrow$ (ii) in the theorem 3.2 is the next proposition also true for any β. (We denote $v^+ = \max(v,0)$).

PROPOSITION 3.3. Let $u_0, \hat{u}_0 \in D(A)$ satisfying:

$$u_0 = \hat{u}_0 \quad \text{and} \quad \frac{\partial u_0}{\partial n} = \frac{\partial \hat{u}_0}{\partial n} \quad \text{a.e. on } \Gamma .$$

Then if $u = u_\lambda(\cdot,u_0,0)$ and $\hat{u} = \hat{u}_\lambda(\cdot,\hat{u}_0,0)$:

$$\|[(u - e^{-t/\lambda}u_0) - (\hat{u} - e^{-t/\lambda}\hat{u}_0)]^+\|_{L^2(\Omega)} \leq (\sinh \frac{t}{\lambda})\|(u_0 - \hat{u}_0)^+\|_{L^2(\Omega)} .$$

In particular:

$$\|(u - e^{-t/\lambda}u_0) - (\hat{u} - e^{-t/\lambda}\hat{u}_0)\|_{L^2(\Omega)} \leq (\sinh \frac{t}{\lambda})\|u_0 - \hat{u}_0\|_{L^2(\Omega)} ,$$

$$u_0 \leq \hat{u}_0 \Rightarrow u(t) \leq \hat{u}(t) \qquad \forall t$$

$$\|u(t) - \hat{u}(t)\|_{L^2(\Omega)} \leq (\cosh \frac{t}{\lambda})\|u_0 - \hat{u}_0\|_{L^2(\Omega)} .$$

Remark 3.5. As in remark 3 of the section 1, let us denote $S_\lambda(t)u_0 = u_\lambda(t,u_0,0)$ the "pseudoparabolic semigroup" associated with A. For any $[h,k] \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ with $k \in \beta(h)$ a.e. on $\partial\Omega$, we set:

$$D_{h,k} = \{u_0 \in D(A) \; ; \; u_0\Big|_{\partial\Omega} = h \quad , \quad -\frac{\partial u_0}{\partial n}\Big|_{\partial\Omega} = k\} .$$

Then the proposition 3.3 says that the restriction of $S_\lambda(t)$ to any $D_{h,k}$ is a Lipschitz-continuous application from $D_{h,k}$ into $D(A)$. Since $D_{h,k}$ is dense in $L^2(\Omega)$, any such restriction can be continuously extended to $\overline{D(A)} = L^2(\Omega)$. In general for nonlinear β such extensions might be different as shown by the corollary 3.1 and their value at 0 might be different from $S_\lambda(t)0$.

Moreover, this proposition says that the restriction of $S_\lambda(t)$ to each $D_{h,k}$ is nondecreasing even though such a property need not be true for $S_\lambda(t)$ itself even if β satisfies condition (ii) of theorem 3.2.

The proof of these results employs an integrated form of (E_2) as given by the following lemma:

Lemma 3.1. Let u be the solution of (E_2), then $v = u - e^{-t/\lambda}u_0$ is the solution of

$$
(*) \qquad \begin{cases} v - \lambda\Delta v = \frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}(v(\sigma) + \lambda f(\sigma))d\sigma + \frac{t}{\lambda}e^{-t/\lambda}u_0 \\ -\frac{\partial v}{\partial n} \in \beta(v + e^{-t/\lambda}u_0) - e^{-t/\lambda}k_0 \quad \text{a.e. on } \Gamma , \end{cases}
$$

where $k_0 = -\frac{\partial u_0}{\partial n} \in \beta(u_0)$.

Proof of lemma 3.1.

We write (E_2) as

$$
\frac{\partial}{\partial t}(u - \lambda\Delta u) + \frac{1}{\lambda}(u - \lambda\Delta u) = \frac{1}{\lambda}(u + \lambda f) \quad ,
$$

multiply by $e^{t/\lambda}$ and integrate over $(0,t)$ to obtain:

$$
u(t) - \lambda\Delta u(t) = e^{-t/\lambda}(u_0 - \lambda\Delta u_0) + \frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}(u(\sigma) + \lambda f(\sigma))d\sigma \quad .
$$

By setting $u(\sigma) = v(\sigma) + e^{-\sigma/\lambda}u_0$ the lemma follows.

Proof of the proposition 3.3.

For $(h,k) \in H^{3/2}(\Omega) \times H^{1/2}(\Gamma)$ with $k \in \beta(h)$ a.e. on Γ, given $t \in [0,T]$ fixed, consider:

$$A^t = \{[v,w] \in L^2(\Omega) \times L^2(\Omega) \; ; \; v \in H^2(\Omega) \quad , \quad -\frac{\partial v}{\partial n} \in \beta(v + e^{-t/\lambda}h) - e^{-t/\lambda}k$$

$$\text{a.e. on } \Gamma \quad , \quad w = -\Delta v\} \quad .$$

Let $u_0 \in D(A)$ with $u_0 = h$ and $-\frac{\partial u_0}{\partial n} = k$ a.e. on Γ (such an u_0 always exists - see [16] -). Then:

$$[v,w] \in A^t \Leftrightarrow [v + e^{-t/\lambda}u_0 \quad , \quad w - e^{-t/\lambda}\Delta u_0] \in A \quad .$$

Hence A^t is maximal monotone in $L^2(\Omega)$ and v satisfies (*) *if and only if*:

$$(**) \qquad v = (I + \lambda A^t)^{-1}[\frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}(v(\sigma) + \lambda f(\sigma))d\sigma + \frac{t}{\lambda}e^{-t/\lambda}u_0] \quad .$$

Since: $\forall v, \hat{v} \in D(A^t)$

$$\int_\Omega (v - \hat{v})^+(-\Delta(v - \hat{v})) = \int_\Gamma (v - \hat{v})^+(\beta(v + e^{-t/\lambda}h) - \beta(\hat{v} + e^{-t/\lambda}h)) \geq 0 \quad ,$$

we see that

$$\forall v, \hat{v} \in D(A^t) \quad , \int_\Omega (v - \hat{v})^+(A^t v - A^t\hat{v}) \geq 0 \quad .$$

Therefore, if $v = (I + \lambda A^t)^{-1}\theta$ and $\hat{v} = (I + \lambda A^t)^{-1}\hat{\theta}$, multiplying

$$v - \hat{v} + \lambda(A^t v - A^t \hat{v}) = \theta - \hat{\theta}$$

by $(v - \hat{v})^+$, we deduce:

$$\|(v - \hat{v})^+\|_{L^2(\Omega)} \leq \|(\theta - \hat{\theta})^+\|_{L^2(\Omega)} \quad .$$

Now if u_0 and $\hat{u}_0$ satisfy $u_0 = \hat{u}_0$ and $\frac{\partial u_0}{\partial n} = \frac{\partial \hat{u}_0}{\partial n}$ a.e. on Γ, and if $v(t) = u_\lambda(t,u_0,0) - e^{-t/\lambda}u_0$, $\hat{v}(t) = u_\lambda(t,\hat{u}_0,0) - e^{-t/\lambda}\hat{u}_0$, by lemma 3.1, we have:

$$v = (I + \lambda A^t)^{-1}[\frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}v(\sigma)d\sigma + \frac{t}{\lambda}e^{-t/\lambda}u_0]$$

$$\hat{v} = (I + \lambda A^t)^{-1}[\frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}\hat{v}(\sigma)d\sigma + \frac{t}{\lambda}e^{-t/\lambda}u_0] \quad ,$$

for the $\underline{\text{same}}$ operator A^t. Hence, from the above inequality, we obtain

$$\|(v-\hat{v})^+\|_{L^2(\Omega)}(t) \le \frac{1}{\lambda}\int_0^t \|(v-\hat{v})^+\|_{L^2(\Omega)}(\sigma)d\sigma + \frac{t}{\lambda}e^{-t/\lambda}\|(u_0-\hat{u}_0)^+\|_{L^2(\Omega)} .$$

Since $\frac{t}{\lambda}e^{-t/\lambda} \le 1-e^{-t/\lambda}$, this proves that $\|(v-\hat{v})^+\|_{L^2}(t)$ is majorized by $\phi(t) = (\sinh\frac{t}{\lambda}).\|(u_0-\hat{u}_0)^+\|_{L^2(\Omega)}$, the solution of

$$\phi(t) = \frac{1}{\lambda}\int_0^t \phi(\sigma)d\sigma + (1-e^{-t/\lambda})\|(u_0-\hat{u}_0)^+\|_{L^2} .$$

This gives the first inequality. The others follow.

<u>Proof of (i) ⇒ (ii) in theorem 3.2.</u>

Let $u_0 \in D(A)$ be fixed with $-\frac{\partial u_0}{\partial n} = k_0 \in \beta^0(u_0)$, $u_0 \ge 0$ and let (φ_n) be a nondecreasing sequence of nonnegative functions in $C_0^\infty(\Omega)$ converging to 1 a.e. in Ω. Then setting $u_0^n = u_0(1-\varphi_n)$, u_0^n converges to u_0 in $L^2(\Omega)$, and for all $n \in \mathbb{N}$,

$$u_0^n = u_0 \quad \text{and} \quad -\frac{\partial u_0^n}{\partial n} = -\frac{\partial u_0}{\partial n} = k_0 \quad \text{on } \Gamma .$$

Set $u^n = u_\lambda(\cdot,u_0^n,0)$ and $v^n = u^n - e^{-t/\lambda}u_0^n$. Then by the proposition 3.3, v^n converges in $C([0,T];L^2(\Omega))$ to v_∞ and by (**) (see the proof of proposition 3.3):

$$v_\infty(t) = (I+\lambda A^t)^{-1}\left[\frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}v_\infty(\sigma)d\sigma\right]$$

where A^t is the operator "associated" with u_0; that is:

$$(*)_\infty \qquad \begin{cases} v_\infty(t) - \lambda\Delta v_\infty(t) = \frac{1}{\lambda}\int_0^t e^{(\sigma-t)/\lambda}v_\infty(\sigma)d\sigma \\ -\frac{\partial v_\infty(t)}{\partial n} \in \beta(v_\infty(t) + e^{-t/\lambda}u_0) - e^{-t/\lambda}k_0 \quad \text{a.e. on } \partial\Omega . \end{cases}$$

Now, if we assume (i), $u^n(t) \geq 0$ $\forall n$, $\forall t$ and since u_0^n converges to 0 in $L^2(\Omega)$, u^n converges to v_∞ in $C([0,T];L^2(\Omega))$. Therefore:

$$\forall t \in [0,T] \quad , \quad v_\infty(t) \geq 0 \quad .$$

Integrating $(*)_\infty$ over Ω, we have:

$$\int_\Omega v_\infty(t) + \lambda \int_\Gamma k(t) - e^{-t/\lambda} k_0 = \frac{1}{\lambda} \int_\Omega \int_0^t e^{(\sigma-t)/\lambda} v_\infty(\sigma) d\sigma \quad ,$$

with $k(t) \in \beta(v_\infty(t) + e^{-t/\lambda} u_0)$ (and $k_0 \in \beta^0(u_0)$). Since $v_\infty(t)$ converges in $L^2(\Omega)$ to 0 when t goes to 0, we deduce (assuming $u_0 > 0$ on Γ):

$$\limsup_{t \to 0^+} \frac{1}{t} \int_\Gamma \left\{ \beta^0(e^{-t/\lambda} u_0) - e^{-t/\lambda} \beta^0(u_0) \right\} \leq 0 \quad .$$

Now, let $r \in]0,\infty[\cap D(\beta)$. There exists $u_0 \in D(A)$ with $u_0 \geq 0$, $u_0|_\Gamma = r$ and $-\frac{\partial u_0}{\partial n} \in \beta^0(r)$ (see [17]). Applying the inequality above with this choice of u_0:

$$\forall r > 0 \quad , \quad \limsup_{t \to 0^+} \frac{1}{t} [\beta^0(e^{-t/\lambda} r) - e^{-t/\lambda} \beta^0(r)] \leq 0 \quad ,$$

and

$$\forall r > 0 \quad , \quad \limsup_{t \to 0^+} \frac{1}{t} \left[\frac{\beta^0(e^{-t/\lambda} r)}{e^{-t/\lambda} r} - \frac{\beta^0(r)}{r} \right] \leq 0 \quad .$$

This and the next lemma applied to $g(r) = \frac{\beta^0(r)}{r}$ imply that $r \to \frac{\beta^0(r)}{r}$ is nondecreasing on $]0,\infty[\cap D(\beta)$.

<u>Lemma 3.2</u>. Let $g : (0,a) \to \mathbb{R}$ possessing at each $r \in (0,a)$ left-limit $g(r^-)$ and right-limit $g(r^+)$ satisfying

$$\forall r \in (0,a) \qquad g(r) \leq g(r^+) \quad .$$

Suppose:

$$\forall r \in (0,a) \quad , \quad \liminf_{h\to 0^+} \frac{g(r-h)-g(r)}{h} \le 0 \quad .$$

Then g is nondecreasing on $(0,a)$.

Proof of lemma 3.2.

Set $r_0 = \inf\{r \in [r_1,r_2] \; ; \; g(r) \le g(r_2) + \varepsilon(r_2 - r)\}$ where ε is an arbitrary fixed positive number. To prove the lemma we will show that $r_0 = r_1$. If $r_1 < r_0 \le r_1$, then by definition of r_0 and $g(r_0) \le g(r_0^+)$:

$$g(r_0) \le g(r_2) + \varepsilon(r_2 - r_0) \quad .$$

Setting $\alpha(h) = \dfrac{g(r_0-h) - g(r_0)}{h}$, above yields:

$$g(r_0 - h) \le g(r_2) + \varepsilon(r_2 - (r_0 - h)) + h(\alpha(h) - \varepsilon) \quad .$$

Since $\liminf_{h\to 0^+} \alpha(h) \le 0$, there exists $h > 0$ such that $r_0 - h \in [r_1,r_2]$ and

$$g(r_0 - h) \le g(r_2) + \varepsilon(r_2 - (r_0 - h)) \quad .$$

This contradicts the definition of r_0. Hence $r_0 = r_1$.

As a corollary of the proof of (i) $\Rightarrow$ (ii) in theorem 3.2, we obtain the following: let $S_\lambda(t)$ denote the pseudoparabolic semigroup associated with A and defined from $D(A)$ into itself (see remark 3 in section 1); suppose β is differentiable at the origin (for simplicity). Then:

Corollary 3.1. The semigroup $S_\lambda(t)$ can be continuously extended to $L^2(\Omega)$ for all $t \in [0,\infty[$ if and only if β is linear.

Proof of corollary 3.1.

From the proposition 1.2, $S_\lambda(t)$ is a contraction for any t if β is linear.

Now suppose that $S_\lambda(t)$ can be continously extended to $L^2(\Omega)$. Let $u_0 \in D(A)$ and $u_0^n = u_0(1 - \varphi_n)$ defined as in the proof of (i) $\Rightarrow$ (ii) above. Then, $u^n(t) = S_\lambda(t)u_0^n$ converges to $S_\lambda(t)0 = 0$. But we also know that u^n converges to v_∞ solution of $(*)_\infty$. Hence $v_\infty(t) = 0$ and $\frac{\partial v_\infty(t)}{\partial n} = 0$ for any t. As u_0 is arbitrary in $D(A)$, we deduce:

$$\forall r \in D(\beta) \quad , \quad \forall k \in \beta(r) \quad , \quad \forall t \in [0,\infty[\quad , \quad e^{-t/\lambda}k \in \beta(e^{-t/\lambda}r) \quad .$$

This is also:

$$\forall r \in D(\beta) \quad , \quad \forall \lambda \in]0,1] \quad , \quad \lambda\beta(r) \subset \beta(\lambda r) \quad .$$

This property together with the differentiability in 0 yields the linearity of β.

<u>Proof of (ii) ⇒ (iii) in theorem 3.2.</u>

Suppose $u_0 \geq 0$ and $f \geq 0$; then $v = u - e^{-t/\lambda}u_0$ is a solution of $(*)$ in lemma 3.1. Multiplying this equation by v^-, we obtain:

$$\int_\Omega (v^-)^2(t) + \int_\Omega (\nabla v^-)^2(t) \leq \int_\Gamma v^-(\beta(v + e^{-t/\lambda}u_0) - e^{-t/\lambda}k_0) + \frac{1}{\lambda}\int_\Omega v^-(t)\int_0^t v^-(\sigma)d\sigma \quad .$$

By (ii), on the set $[v < 0] \cap [u_0 > 0]$:

$$\beta(v + e^{-t/\lambda}u_0) \leq \beta^0(e^{-t/\lambda}u_0) \leq e^{-t/\lambda}\beta^0(u_0) \leq e^{-t/\lambda}k_0$$

and on the set $[v < 0] \cap [u_0 = 0]$

$$\beta(v + e^{-t/\lambda}u_0) \leq \beta(0^-) \leq e^{-t/\lambda}k_0 \quad (k_0 \in \beta(0)) \quad .$$

Therefore

$$\int_\Omega (v^-)^2(t) \leq \frac{1}{\lambda}\int_\Omega v^-(t)\int_0^t v^-(\sigma)d\sigma \leq \frac{1}{\lambda}\|v^-(t)\|_{L^2(\Omega)} \cdot \int_0^t \|v^-(\sigma)\|_{L^2(\Omega)} d\sigma \quad ,$$

By Gronwall lemma, $\forall t$, $v^-(t) = 0$ and hence $v(t) \geq 0$.

If either $f \not\equiv 0$ or $u_0 \not\equiv 0$, as $I - \lambda\Delta$ satisfies a strong maximum principle, from equation $(*)$ we deduce:

$$\forall t > 0 \quad , \quad v(t) > 0 \quad .$$

This proves (i) and (iv)

As the implication (iii) ⇒ (i) is trivial, the proof of the theorem 3.2 is complete.

Proof of Proposition 3.2.

Let us consider H the solution of:

$$\begin{cases} \frac{\partial}{\partial t}(H - \lambda\Delta H) - \Delta H = f \\ -\frac{\partial H}{\partial n} \in \gamma(H) \quad , \quad H(0) = u_0 - G \quad , \end{cases}$$

(where G is defined in the proposition) and γ the maximal monotone graph defined by:

$$\gamma(r) = \begin{cases} \phi & r > 0 \\ [0,\infty] & r = 0 \\ 0 & r < 0 \quad . \end{cases}$$

The initial data $u_0 - G$ belongs to $H^2(\Omega)$; moreover as $u_0 - G = 0$ on Γ and $u_0 - G \geq 0$ on Ω, $-\frac{\partial}{\partial n}(u_0 - G) \geq 0$ on Γ; hence $-\frac{\partial}{\partial n}(u_0 - G) \in \gamma(0) = \gamma(u_0 - G)$ a.e. on Γ and the problem above can be solved for H.

Since γ satisfies the condition (ii) of the theorem 3.2 and since $H(0) \geq 0$, $f \geq 0$, for all $t \in [0,T]$, $H(t) \geq 0$. We will show that $u(t) = u_\lambda(t,u_0,f) \geq H(t)$ for all $t \in [0,T]$.

Set $w = u - H$ and multiply by $(w - \lambda\Delta w)^-$ the difference of the equations defining u and H to get:

$$\int_\Omega (w - \lambda\Delta w)^- \frac{\partial}{\partial t}(w - \lambda\Delta w) + \int_\Omega (w - \lambda\Delta w)^-(-\Delta w) = 0 \quad .$$

Since $\forall a,b \in \mathbb{R}$ $(a + b)^- b \leq a^- b$, above yields:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega ((w - \lambda\Delta w)^-)^2(t) \leq -\int_\Omega w^-\Delta w = \ldots$$

$$\ldots = -\int_\Gamma w^- \frac{\partial w}{\partial n} - \int_\Omega |\nabla w^-|^2 = \int_\Gamma w^-(k - \hat{k}) - \int_\Omega |\nabla w^-|^2 \leq -\int_\Omega |\nabla w^-|^2 \leq 0 \quad .$$

Here we used the fact that $\hat{k} \in \gamma(H)$ is nonnegative and $h \in \beta(u)$ is nonpositive on the set $[w < 0] = [u < H]$.

From the definition of w and H, we have $w(0) - \lambda\Delta w(0) = 0$, so that above implies $w(t) - \lambda\Delta w(t) \geq 0$ for all $t \in [0,T]$. This in turn implies $w(t) \geq 0$ and $u(t) \geq H(t) \geq 0$ for all $t \in [0,T]$.

Remark 3.6. The idea of the proof of $u \geq H$ is the following: if two operators A_1 and A_2 are comparable (in some sense) and if the "thermodynamic" data $u_i(0) + \lambda A_i u_i(0)$ can be compared, then also the respective solutions u_1 and u_2 can be compared. Here we used the fact that $\beta \leq \gamma$.

C) Case (c): "$A = -\Delta + \beta$"

Here we denote by A the m-accretive operator defined in (c), on $L^2(\Omega)$, (Ω bounded). For any $[u_0,w_0] \in A$ and $f \in L^1(0,T;L^2(\Omega))$, there exists a unique solution of

$$(E_3) \qquad \begin{cases} \dfrac{\partial}{\partial t}\,(u + \lambda(-\Delta u + \beta u)) - \Delta u + \beta u \ni f \ , \\ u(0) = u_0 \quad , \quad w_0 \in -\Delta u(0) + \beta u(0) \ . \end{cases}$$

We denote by $u_\lambda(\cdot,u_0,w_0,f)$ such a solution.

Theorem 3.3. Suppose that either

(i) $D(\beta) \subset [0,\infty[$, or

(ii) the map $\left\{ r \in \,]0,\infty[\, \cap D(\beta) \to \lambda \dfrac{\beta^0(r)}{r} + \ln r \right\}$ is nondecreasing .

Then

$$\text{(iii)} \quad \begin{cases} \forall u_0 \in D(A) \quad , \quad \forall w_0 \in Au(0) \\ (u_0 \geq 0 \quad , \quad f \geq 0) \Rightarrow (u_\lambda(t,u_0,w_0,f) \geq 0 \quad \forall t \in [0,T]) \ . \end{cases}$$

In particular if $\beta(\cdot)$ satisfies (ii) then

$$u_\lambda(t,u_0,w_0,f) \geq e^{-t/\lambda} u_0 \ .$$

If in addition the map $\left\{ r \in [0,\infty[\, \cap D(\beta) \to \beta(r) \right\}$ is single valued and continuously differentiable, then (iii) $\Rightarrow$ (ii).

Remark 3.7. (a) Condition (ii) is similar (but weaker) to the condition (ii) of the previous case B.

(b) Here we could also replace $-\Delta$ with any linear operator L in $L^2(\Omega)$, satisfying the maximum principle (M_0) in 3.1.

The proof of the theorem employs the following integrated form of (E_3). We denote by $h_0 \in \beta u_0$ that element out of the set βu_0 for which $w_0 = -\Delta u_0 + h_0$.

Lemma. Let u be the solution of (E_3). Then $v = u - e^{-t/\lambda} u_0$ satisfies

$$(3.6) \qquad v - \lambda\Delta v + \lambda\gamma(t,v) \ni \frac{1}{\lambda}\int_0^t e^{(s-t)/\lambda}(v(s) + \lambda f(s))ds \quad ,$$

where $\gamma(t,v) = \beta(v + e^{-t/\lambda}u_0) - e^{-t/\lambda}\left(h_0 + \frac{t}{\lambda^2}u_0\right)$.

Proof of the Lemma.

We write (E_3) in the form

$$\frac{\partial}{\partial t}[u + \lambda(-\Delta u + \beta u)] + \frac{1}{\lambda}[u + \lambda(-\Delta u + \beta u)] \ni \frac{1}{\lambda}(u + \lambda f) \quad ,$$

multiply by $e^{t/\lambda}$ and integrate over $(0,t)$ to obtain:

$$u(t) + \lambda(-\Delta u(t) + \beta u(t)) \ni e^{-t/\lambda}(u_0 + \lambda(-\Delta u_0 + h_0)) + $$

$$+ \frac{1}{\lambda}\int_0^t e^{(s-t)/\lambda}(u(s) + \lambda f(s))ds \quad .$$

Finally writing $u(s) = v(s) + e^{-s/\lambda}u_0$ in the integral gives (3.6).

Proof of Theorem 3.3 (sufficient condition):

If $D(\beta) \subset [0,\infty[$ the statement is trivial. Assume that (ii) holds. Then

$$\forall r > 0 \quad \frac{\beta^0(e^{-t/\lambda}r)}{e^{-t/\lambda}r} + \frac{1}{\lambda}\ln(e^{-t/\lambda}r) \le \frac{\beta^0(r)}{r} + \frac{1}{\lambda}\ln r \quad ,$$

which can be rewritten as

$$\forall r > 0 \quad \beta^0(e^{-t/\lambda} r) - \frac{t}{\lambda^2} r e^{-t/\lambda} \le e^{-t/\lambda} \beta^0(r) \quad .$$

Therefore if v is a solution of (3.6) with $u_0 \geq 0$, then

$$\int_\Omega \operatorname{sgn}^- v \cdot \gamma(t,v) \le 0 \quad , \quad \text{a.e.} \quad t \in [0,T] \quad .$$

Now we multiply (3.6) by $\operatorname{sgn}^- v$ and take in account that

$$\int_\Omega \operatorname{sgn}^- v(-\Delta v) \le 0$$

and the nonnegativity of $f(\cdot)$, to obtain

$$\int_\Omega v^-(t) + \frac{1}{\lambda} \int_\Omega \operatorname{sgn}^- v(t) \int_0^t e^{(s-t)/\lambda} v(s)ds \le 0 \quad .$$

Writing $v(s) = v^+(s) - v^-(s)$ and majorizing the second integral gives

$$\int_\Omega v^-(t)dx \le \frac{1}{\lambda} \int_0^t \int_\Omega v^-(s)dxds \quad .$$

By the Gronwall inequality, this implies $v^-(t) = 0$ a.e. in $\Omega \times [0,T]$.

Proof of Theorem 3.3 (necessary condition):

If $\beta(\cdot)$ is continuously differentiable, we will show that (iii) $\Rightarrow$ (ii) by exploiting the arbitrariness of $u_0 \geq 0$ (to assume $f \equiv 0$ will be sufficient).

We first prove the result under the extra assumption that β is continuously differentiable on $[0,\infty)$ and $\beta'(r)$ is uniformly bounded on $[0,\infty)$.

Let $B(2\rho)$ be a ball of radius 2ρ, $(\rho > 0)$ contained in Ω and let $r > 0$ be fixed but arbitrary.

Consider a sequence of nonnegative $C_0^\infty(\Omega)$ functions u_0^n, $n \in \mathbb{N}$ such that

(a) $\operatorname{supp} u_0^n \subset B(2\rho) \quad , \quad \forall n \in \mathbb{N}$

(b) $u_0^n(x) = r \quad , \quad x \in B(\rho) \quad , \quad \forall n \in \mathbb{N}$

(c) $u_0^n \to r\chi[B(\rho)]$ in $L^2(\Omega)$,

where $\chi[B(\rho)]$ denotes the characteristic function of the ball $B(\rho)$.

We denote $u^n(\cdot) = u_\lambda(\cdot, u_0^n, 0) \geq 0$.

Next we construct "test-functions" φ_n as follows. Let $g \in L^2(\Omega)$, $g \geq 0$ be not identically zero in Ω, such that

$$\text{supp } g \subset \Omega \setminus B(2\rho)$$

and let $\varphi_n(t)$ be the unique solution of

$$(3.7) \qquad \begin{cases} [1 + \lambda\beta'(u^n(t))]\varphi_n(t) - \lambda\Delta\varphi_n(t) = g & \text{in } \Omega \\ \gamma(\varphi_n(t)) = 0 & \text{on } \Gamma \end{cases}$$

We remark that $\varphi_n(t) \in C([0,T];L^2(\Omega))$ since $\beta'(\cdot)$ is continuous on $[0,\infty)$ and $u^n \in C([0,T];L^2(\Omega))$. Since $g \not\equiv 0$ by the strong maximum principle

$$\varphi_n(x,t) > 0 \quad \text{in } \Omega \qquad \forall t \in [0,T] \text{ and } \forall n \in \mathbb{N} .$$

Moreover it is easy to verify that $\varphi_n(0) \to \varphi_\infty(0)$ in $L^2(\Omega)$, where $\varphi_\infty > 0$ is the solution of

$$\begin{cases} [1 + \lambda\beta'(r\chi[B(\rho)])]\varphi_\infty - \lambda\Delta\varphi_\infty = g \\ \gamma(\varphi_\infty) = 0 \end{cases} .$$

We multiply (E_3) (written with $f(t) \equiv 0$ and initial datum u_0^n) by φ_n and integrate by parts, to obtain

$$\int_\Omega u_t^n(\varphi_n + \lambda\beta'(u^n)\varphi_n - \lambda\Delta\varphi_n)dx = \int_\Omega \{u\Delta\varphi - \beta(u)\varphi\}dx =$$

$$= \frac{1}{\lambda}\int_\Omega \{u^n[(1 + \lambda\beta'(u^n))\varphi_n - g] - \lambda\beta(u^n)\varphi_n\}dx .$$

Here we used the fact that under the stated assumption on $\beta(\cdot)$, (E_3) can be written as

$$\frac{d}{dt}(u - \lambda\Delta u) + \beta'(u)u_t - \Delta u + \beta(u) = 0 \quad ,$$

whose pointwise meaning is easy to justify.

By (3.7) above can be rewritten as

$$\int_\Omega u_t^n g \, dx = \frac{1}{\lambda}\int_\Omega [u^n(1 + \lambda\beta'(u^n)) - \lambda\beta(u^n)]\varphi_n \, dx - \frac{1}{\lambda}\int_\Omega u^n g \, dx \quad . \tag{3.8}$$

We observe that

$$\int_0^t \int_\Omega u_t^n g \, dx d\tau = \int_\Omega u^n(t) g \, dx - \int_\Omega u_0^n(x) g \, dx \geq 0$$

because $u^n \geq 0$, $g \geq 0$, and the particular choice of g.

Therefore (3.3), and $u^n \geq 0$ $g \geq 0$ imply

$$0 \leq \int_0^t \int_\Omega [u^n(1 + \lambda\beta'(u^n)) - \lambda\beta(u^n)]\varphi_n \, dx d\tau \qquad \forall n \in \mathbb{N} \quad .$$

Dividing by t and letting $t \downarrow 0$ we obtain

$$0 \leq \int_\Omega [u_0^n(1 + \lambda\beta'(u_0^n)) - \lambda\beta(u_0^n)]\varphi_n(0) \, dx \quad , \quad \forall n \in \mathbb{N} \quad .$$

As $n \to \infty$ $u_0^n \to r\chi[B(\rho)]$ in $L^2(\Omega)$, $\varphi_n \to \varphi_\infty > 0$ in $L^2(\Omega)$ and for a suitable subsequence $\beta'(u_0^n) \to \beta'(r\chi[B(\rho)])$.

Hence we can pass to the limit under integral as $n \to \infty$, to obtain

$$[r(1 + \lambda\beta'(r)) - \lambda\beta(r)] \int_{B(\rho)} \varphi_\infty \, dx \geq 0$$

Since $\varphi_\infty > 0$ in Ω, this in turn gives

$$\frac{\lambda\beta(r)}{r} \leq 1 + \lambda\beta'(r) \quad , \quad r \in \mathbb{R}^+$$

i.e.

$$\frac{d}{dr}\left(\frac{\lambda\beta(r)}{r} + \ln r\right) \geq 0 \quad , \quad r \in \mathbb{R}^+ \quad .$$

This concludes the proof of the theorem, in the case of $\beta'(r)$ uniformly bounded $\forall r \in \mathbb{R}^+$.

Suppose now that $r \in [0,\infty[\cap D(\beta) \to \beta^0(r)$ is continuously differentiable.

Let $r \in D(\beta)$ be selected and let $\{u_0^n\}$ be constructed as in the first part of the proof.

Let $\eta > 0$ such that $r + \eta \in D(\beta)$ and denote by γ a continuously differentiable maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that

(i) $\gamma(s) = \beta(s) \quad s \leq r + \eta$

(ii) $\gamma \leq \beta^0$ on $D(\beta)$

(iii) $\gamma'(r)$ is uniformly bounded $\forall r \in [0,\infty)$.

If we denote by $u_\lambda^\gamma(\cdot,u_0^n,0)$ the solution of (E_3) with β replaced by γ, then by virtue of the Remark 3.6

$$u_\lambda^\gamma(\cdot,u_0^n,0) \geq u_\lambda(\cdot,u_0^n,0) \geq 0 \quad .$$

Hence the argument can be repeated on the $u_\lambda^\gamma(\cdot,u_0^n,0)$ to yield the result.

Remark 3.8. We do not expect condition (ii) to be necessary without the assumption of continuous differentiability on $\beta(\cdot)$. We saw that (iii) is satisfied when $D(\beta) \subset [0,\infty)$ regardless of the behavior of $\beta(\cdot)$ in its domain (in particular when (ii) is violated). If we assume $\beta(\cdot)$ differentiable at 0, then $0 \in \text{Int } D(\beta)$, hence we are not in the previous situation. It would be of interest to know whether only the assumption of differentiability at the origin suffices in order for (ii) to be necessary.

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For an m-accretive operator A in a Banach space X, we investigate the invariance of the solution of $\frac{d}{dt}(u + \lambda Au) + Au \ni 0$ with respect to a convex cone, under the assumption that the resolvents of A leave invariant the cone.
If in particular X is a function space and above represents a partial differential equation, necessary and sufficient conditions are given on the boundary data to insure the nonnegativity of the solution.

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